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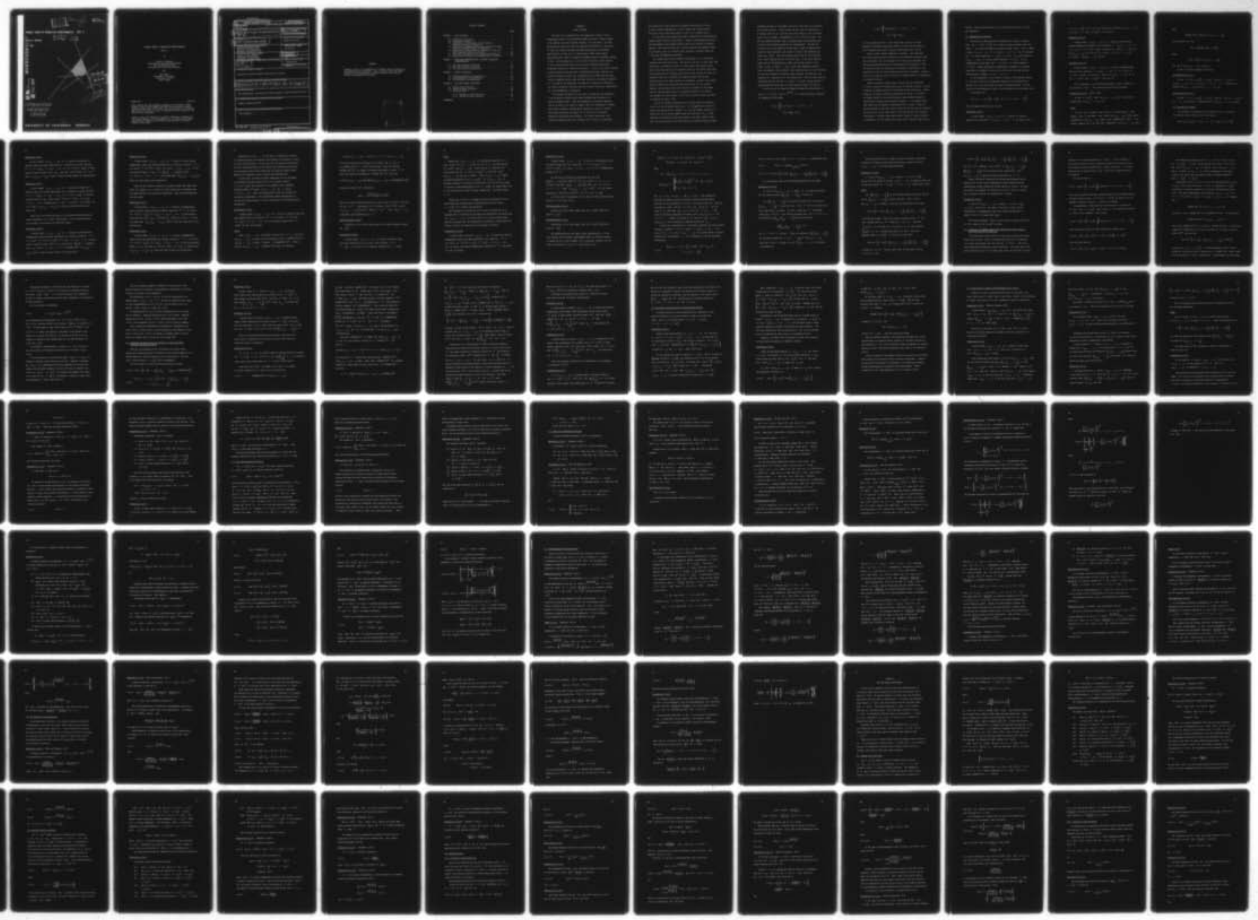

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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES: PART II

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by

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and

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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES†

PART II

by

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ABSTRACT

Chapters 3 thru 6 of a monograph on a Dynamic Theory of Production Correspondences are presented. Laws of Return, Functional Representation of Correspondences, Special Structures and Cost and Revenue Functionals are discussed.

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CHAPTER 3

LAWS OF RETURN

The laws to be considered here are suggested by Turgot's law of diminishing returns, on the intensive margin so called. Loosely worded, in the context of static economic analysis, this law states: as equal quantities of capital and labor are applied successively to a given plot of land, the output resulting from these applications will increase monotonically at first up to a certain point, after which further applications will result in steadily decreasing product increments tending to zero. This expression of a law of return postulates smooth behavior in fine structure which cannot be justified by generally applicable properties (axioms) for production structure. As the law is often reflected by the production functions used in neoclassical economic analysis, unbounded increase in output rate is possible for any bound upon the input rates of essential factors when the other factors are increased indefinitely. As found in the literature such behavior is merely that implied by the concave production functions used and it has no particular significance for the phenomenology of production.

The original motivation for such laws of return was the recognition of the restraint to agricultural output arising from the scarcity of land as a primary factor. What then appeared to be obvious regarding land as an input is no longer so simple, since the techniques of farming have greatly expanded, yielding great increases in output to nourish even larger populations of humans, but at the price of increased dependence upon manufactured products. For modern technology, both farming and manufacturing, the limiting role of inputs is considerably

more complicated, where some may be complete substitutes for others as well as being complementary to still others, and institutionally any factor may be restricted whether or not it is a primary factor.

As one views such possibilities today, limitations of world resources present serious limits on some inputs. Energy is increasingly more dear and environmental aims present serious limits on production to preserve habitat. Thus laws of return are a central issue in the economic theory of production, both as to output rates, and span of possible output as in the case of exhaustible resources.

For steady state (static) models of production, i.e., the case of constant input and output rates, (see Section 2.6) laws of return have been established in (Shephard, 1970:b) for technologies with single output. There, it has been shown for an *essential* subset of the factors that there exist bounds upon the constant input rate of these factors such that the constant output rate obtainable is bounded no matter how much the constant rates of the other factors are increased. Further it was shown under the axioms for such structures (see Section 2.6), by counter-example, that constant output rate need not be bounded for all bounds upon the constant input rates of an essential subset of the factors. The extension of these results to the case of multiple outputs was made in (Shephard and Färe, 1974).

The law of return so expressed for the static model of production is one of a law of bounded output rate. It is suggested for input and output rates which are not constant, i.e., for the dynamic structure of production, that a law of bounded output rate may hold, i.e., if time histories for *essential* factors are subject to an upper bound on input rate, the related output rate histories will be bounded in some way under

unlimited increase in the maximal time rate of the input rate histories of the other factors. In this connection the axiom on disposability of outputs plays a role, and there are three cases of a law of bounded ouput rate to consider, corresponding to $\mathbb{L}.6$, $\mathbb{L}.6S$ and $\mathbb{L}.6SS$. One may also express different laws of bounded output rate depending upon the input rate histories permitted for consideration. The most all inclusive case is one where the entire space $(L_\infty)^n_+$ is permitted for vectors of input histories, and vectors of output rate histories are taken from $(L_\infty)^m_+$. Here the stronger axioms $\mathbb{E}.S$ and $\mathbb{E}$ are needed, with a weak* topology used in the latter case. On the other hand one might restrict consideration of output rate history vectors to the subset $(\tilde{L}_\infty)^m_+$ of vectors u from $(L_\infty)^m_+$ with component histories which are summable, since infinite total production may not be of interest, but the time extension of positive output rate may be permitted to be unbounded as a possibility for time substitution. Then the same norms and topologies used for the case $x \varepsilon (L_\infty)^n_+$, $u \varepsilon (L_\infty)^m_+$, may be applied to express somewhat weaker laws of return using the axioms $\tilde{\mathbb{E}}.S$ and $\tilde{\mathbb{E}}$.

A third case is one where the dynamic production correspondence is taken as $u \varepsilon (L_1)^m_+ \rightarrow \mathbb{L}_1(u) \varepsilon 2^{(L_1)^n_+}$, i.e., input and output histories are summable with the norms

$$||u_i|| = \int_0^\infty |u_i(t)| d\nu_i(t) \qquad i \varepsilon \{1,2, \ldots, m\}$$

$$||u|| = \underset{i}{\mathrm{Max}} \{||u_i||\}$$

$$||x_i|| = \int_0^\infty |x_i(t)|d\mu_i(t) \quad , \; i \in \{1,2, \ldots, n\}$$

$$||x|| = \underset{i}{\text{Max}} \; \{||x_i||\}$$

for the L_1-spaces involved. With these norms, the law of return obtained by following an analysis similar to the other two cases is a law of bounded total output, using the weak axioms $\mathbb{E}_1 S$ and $\mathbb{E}_1$.

There is still another dimension to consider. The time spans over which essential input rates may be or are applied positively need not be infinite, that is the support of an input may be bounded, and unbounded time substitutions for resources may not be permitted. Then the question arises how outputs may be limited by limitations on the intervals of time over which essential factors may be applied. Propositions of this type are laws of return for bounded intervals of application of essential factors.

Hence two general types of laws of return will be considered: (1) Laws of Return for bounded input rates of essential factors, (2) Laws of Return for bounded intervals of essential factor application. Before taking up the details concerning these laws of return, it is useful to discuss briefly in the next two sections some concepts related to essentiality of inputs and jointness of outputs. It is possible that some inputs may be completely substituted for by another. Then any bound whatsoever on the input rates of such factors may have no limitation on output rates as the input rates of the other factors are increased indefinitely. Further, some outputs may be linked or jointly involved in production, so that limiting the output rate of one may limit that of

another. These interactions need to be clarified before laws of return are formulated.

3.1 Essentiality of Factors

The factors of production may be essential for outputs in several ways. Let $u = (v,w) \varepsilon (L_\infty)_+^m$ be a vector of output histories with subvector v not empty and possibly the entire output vector u with w empty. A subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of n factors, $(1 \leqq k < n)$, may be essential for: (a) output subvectors $v \varepsilon (L_\infty)_+^\ell$, $1 \leqq \ell \leqq m$, (b) scaled versions $(\theta_1 v_1, \ldots, \theta_\ell v_\ell)$ of a given output subvector $v \varepsilon (L_\infty)_+^\ell$, (c) for homogeneously scaled forms (θv), $\theta \varepsilon (0,+\infty)$, of a given subvector $v \varepsilon (L_\infty)_+^\ell$. These three kinds of essentiality correspond to different situations for disposal of outputs, the third case following from the second merely by taking $\theta_1 = \theta_2 \cdots = \theta_\ell = \theta$.

In the formulations of the axioms a mixture of the disposal allowed by $\mathbb{L}.6S$ and $\mathbb{L}.6$ was not considered, because all such possibilities between $\mathbb{L}.6$, $\mathbb{L}.6S$ and $\mathbb{L}.6SS$ are matters of greater detail. For the purpose of the discussion at hand, it is sufficient to use the axiom $\mathbb{L}.6S$.

As notation, let

$$D(\nu_1,\nu_2, \ldots, \nu_k) = \left\{x \varepsilon (L_\infty)_+^n : x_{\nu_i} = 0 , i \varepsilon \{1,2, \ldots, k\}\right\} .$$

The following definitions will be used.

Definition (3.1-1):

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of n factors is Globally Essential for subvectors $v = (v_1,v_2, \ldots, v_\ell) > 0$ of an output vector

$u = (v,w) \varepsilon (L_\infty)^m_+$, $(1 \leqq \ell \leqq m)$, if and only if $\mathbb{L}(v,w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ for all $v > 0$ and $w \varepsilon (L_\infty)^{n-\ell}_+$, $\mathbb{L}(v,w) \neq \emptyset$.

Definition (3.1-2):

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of n factors is Output Rate History Homogeneously Essential for a subvector $v = (v_1,v_2, \ldots, v_\ell) > 0$, $(1 \leqq \ell \leqq m)$ of $u = (v,w) \varepsilon (L_\infty)^m_+$ if and only if $\mathbb{L}(\theta_1 v_1, \ldots, \theta_\ell v_\ell, w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ for all $\theta_i \varepsilon (0,+\infty)$, $i \varepsilon \{1,2, \ldots, \ell\}$ and $w \varepsilon (L_\infty)^{n-\ell}_+$, $\mathbb{L}(\theta_1 v_1, \ldots, \theta_\ell v_\ell, w) \neq \emptyset$.

Definition (3.1-3):

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of n factors is Homogeneously Essential for a subvector $v = (v_1,v_2, \ldots, v_\ell) > 0$, $(1 \leqq \ell \leqq m)$ of $u = (v,w) \varepsilon (L_\infty)^m_+$ if and only if $\mathbb{L}(\theta v,w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ for all $\theta \varepsilon (0,+\infty)$ and $w \varepsilon (L_\infty)^{n-\ell}_+$, $\mathbb{L}(\theta v,w) \neq \emptyset$.

It is convenient to use the efficient subsets $\mathbb{E}(u)$ instead of the entire set $\mathbb{L}(u)$. The following justifies this procedure.

Proposition (3.1-1): (Färe, 1972)

For any $u \varepsilon (L_\infty)^m_+$, $\mathbb{L}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ if and only if CLOSURE $\mathbb{E}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$.

Proof:

In case $\mathbb{L}(u)$ is empty, $\mathbb{E}(u) \subset \mathbb{L}(u)$ is likewise empty. Hence suppose $\mathbb{L}(u)$ is not empty. Then $\mathbb{L}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ empty implies CLOSURE $\mathbb{E}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ empty, since CLOSURE $\mathbb{E}(u) \subset \mathbb{L}(u)$. Conversely, suppose $\mathbb{L}(u)$ not empty with CLOSURE $\mathbb{E}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ empty.

Then

$$\text{CLOSURE } \mathbb{E}(u) \subset \left((L_\infty)_+^n \sim D(\nu_1,\nu_2, \ldots, \nu_k)\right) .$$

By Proposition (2.2.4-2),

$$\mathbb{L}(u) \subset \left(\text{CLOSURE } \mathbb{E}(u) + (L_\infty)_+^n\right) .$$

Hence

$$\mathbb{L}(u) \subset \left((L_\infty)_+^n \sim D(\nu_1,\nu_2, \ldots, \nu_k)\right)$$

and $\mathbb{L}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty.

By exactly similar arguments one obtains:

Sub-Proposition (3.1-2):

For any $u = (v,w) \varepsilon (L_\infty)_+^m$, $v \varepsilon (L_\infty)_+^\ell$, $\mathbb{L}(\theta_1 v_1, \ldots, \theta_\ell v_\ell, w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$, $\theta_i \varepsilon (0,+\infty)$, $i \varepsilon \{1,2, \ldots, \ell\}$, if and only if CLOSURE $\mathbb{E}(\theta_1 v_1, \ldots, \theta_\ell v_\ell, w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$.

Sub-Proposition (3.1-3):

For any $u \varepsilon (v,w) \varepsilon (L_\infty)_+^m$, $v \varepsilon (L_\infty)_+^\ell$, $\mathbb{L}(\theta v,w) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ for $v > 0$ if and only if CLOSURE $\mathbb{E}(\theta v,w) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty.

3.2 Jointness of Outputs

The strongest, and perhaps the most familiar case, of jointness for multiple output rate histories is one where

$$\mathbb{P}(x) = \left\{u \ \varepsilon \ (L_\infty)_+^m : u = \alpha(x) \cdot u^o , u^o \ \varepsilon \ (L_\infty)_+^m , u_i^o > 0 \ \forall i\right\}$$

and $\alpha(x)$ is a functional $\alpha : x \in (L_\infty)^n_+ \to \alpha(x) \in R_+$, consistent with the axioms $\mathbb{P}.1, \ldots, \mathbb{P}.6$ and u^o satisfies $\mathbb{P}.T.1$, $\mathbb{P}.T.2$. Then each output rate history bears a fixed relation to the others for $t \in [0,+\infty)$. Here the jointness of output is complete, strict, and symmetric with respect to jointness of null output rate histories, i.e., $u_i = 0 \Longleftrightarrow u_j = 0$ for all $j \in (\{1,2, \ldots, m\} \sim i)$, $i \in \{1,2, \ldots, m\}$.

At another extreme there can be no jointness of null output rate histories required, if the correspondence $x \to \mathbb{P}(x)$ satisfies the axiom $\mathbb{P}.6SS$, e.g., where

$$\mathbb{P}(x) = \left\{u \in (L_\infty)^m_+ : u \leqq \alpha(x) \cdot u^o \ , \ u^o \in (L_\infty)^m_+ \ , \ u^o_i > 0 \ \forall i\right\} .$$

For the purpose of the laws of return to be discussed, all special relationships of how output rate histories may vary jointly with respect to each other over time is not of interest. Rather, jointness of null output rate history, i.e., *Null Jointness*, alone is of interest in this context, and this kind of joint relationship need not be symmetric.

Let $u = (v_1,v_2,w)$, where $\{v_1,v_2\} \subset \{u\}$, $\{v_1\} \neq \emptyset$, $\{v_2\} \neq \emptyset$.

<u>Definition (3.2-1)</u>:

A subvector v_1 of output rate histories is Null Joint with a subvector v_2 , $u = (v_1,v_2,w)$, $\{v_1,v_2\} \subset \{u\}$ if and only if $\bar{t}_{v_1} \leqq \bar{t}_{v_2}$.

The following proposition relates null jointness of subvectors of output rate histories to essentiality of input histories.

Proposition (3.2-1):

If a subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of factors is essential for a subvector v_1 of $u = (v_1, v_2, w)$, $\{v_1, v_2\} \subset \{u\}$, and v_1 is null joint with another disjoint subvector v_2 of u , then $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is essential for the subvector (v_1, v_2) of u .

Since $v_1 > 0$ implies $\bar{t}_{v_1} > 0$ and v_1 null joint with v_2 implies $\bar{t}_{v_2} \geqq \bar{t}_{v_1} > 0$, it follows that $v_1 > 0$ implies $v_2 > 0$. Hence

$$\mathbb{L}(v_1, v_2, w \mid v_1 > 0) = \mathbb{L}(v_1, v_2, w \mid v_1 > 0, v_2 > 0) ,$$

and, if $\mathbb{L}(u \mid v_1 > 0) \cap D(\nu_1, \nu_2, \ldots, \nu_n)$ is empty, it follows that $\mathbb{L}(u \mid v_1 > 0, v_2 > 0) \cap D_2(\nu_1, \nu_2, \ldots, \nu_k)$ is empty. Thus $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is essential for (v_1, v_2) if it is essential for v_1 and v_1 is null joint with v_2 .

In this way outputs may be linked concerning essentiality of subsets of the factors. For the purposes of the text to follow, a single subvector v of $u = (v, w) \in (L_\infty)^m_+$, with $\{w\}$ possibly empty, will suffice for consideration of the laws of return.

3.3 Limitation of Output Rates by Input Rates of Essential Factors of Production

A global limitation of output rates is expressed by the following two definitions:

Definition (3.3-1):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors of production is globally input rate weak limitational for a subvector of output histories $v^o > 0$, $v^o \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$, of $u^o = (v^o, w^o) \in (L_\infty)_+^m$ if there exists a positive bound $B(u^o) \in R_{++}$ such that $(v, w^o) \notin \mathbb{P}(x)$ *for* $v \geqq v^o$ when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B(u^o)$ while the other factors are unrestricted.

Definition (3.3-2):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors of production is globally input rate strong limitational for a subvector of output histories $v > 0$, $v \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$, of $u = (v,w) \in (L_\infty)_+^m$ if for each positive bound $B \in R_{++}$ there exists a vector $\bar{u} = (\bar{v}, w)$, $\bar{v} \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$, depending upon B and u, such that $(v,w) \notin \mathbb{P}(x)$ for $v \geqq \bar{v}$ when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B$ while the other factors are unrestricted.

These first two definitions apply to the situation where outputs are freely disposable, i.e., axiom $\mathbb{L}$.6SS applies. For the situation where $\mathbb{L}$.6S is used, the following two definitions apply.

Definition (3.3-3):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors is output history homogeneously input rate weak limitational for scaling a subvector $v^o > 0$ of a vector $u^o = (v^o, w^o) \in (L_\infty)_+^m$, $v^o \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$ if there exists a positive bound $B(u^o) \in R_{++}$ such that for $\left(\left(\theta_1 v_1^o, \ldots, \theta_\ell v_\ell^o\right), w\right) \in \mathbb{P}(x)$, $\theta_i \in (0,+\infty)$, $i \in \{1,2, \ldots, \ell\}$, are bounded when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B(u^o)$ while the other factors are unrestricted.

Definition (3.3-4):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors is output history homogeneously input rate strong limitational for scaling a subvector $v^o > 0$ of a vector $u^o = (v^o, w^o) \in (L_\infty)^m_+$, $v^o \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$, if for each positive bound $B \in R_{++}$, for $\left(\left(\theta_1 v_1^o, \ldots, \theta_\ell v_\ell^o\right), w\right) \in \mathbb{P}(x)$, $\theta_i \in (0,+\infty)$, $i \in \{1,2, \ldots, \ell\}$, are bounded when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B$ while the other factors are unrestricted.

There is still another situation to consider, namely that where only axiom $\mathbb{L}$.6 applies, i.e., a vector of output histories is disposable only by common scaling of the output rate histories. Then the two definitions take the forms:

Definition (3.3-5):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors is homogeneously input rate weak limitational for scaling a subvector $v^o > 0$ of a vector $u^o = (v^o, w^o) \in (L_\infty)^m_+$, $v^o \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$, if there exists a positive bound $B(u^o) \in R_{++}$, such that for $((\theta v^o), w) \in \mathbb{P}(x)$, $\theta \in (0,+\infty)$ is bounded when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B(u^o)$ while the other factors are unrestricted.

Definition (3.3-6):

A proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors is homogeneously input rate strong limitational for scaling a subvector $v^o > 0$ of a vector $u^o = (v^o, w^o) \in (L_\infty)^m_+$, $v^o \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$, if for each positive bound $B \in R_{++}$ such that for $(\theta v^o, w) \in \mathbb{P}(x)$, $\theta \in (0,+\infty)$ is bounded when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B$ while the other factors are unrestricted.

Definitions (3.3-3), ..., (3.3-6) refer to bounding the scaling of individual output rate histories and bounding the scaling of vectors of output rate histories. In the first pair of these four definitions, only the time distributions of the output rate histories are preserved by the scaling, while in the second pair the relation between output histories is preserved as well as their time distributions.

To start with, the case of fewest exceptions will be considered by invoking the stronger axioms $\mathbb{E}.S$, $\mathbb{E}$ for output vectors $u = (v,w) \in (L_\infty)^m_+$ not restricted to be summable, with a globally essential subset of factors $\{\nu_1,\nu_2, \ldots, \nu_k\}$ for $v \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$. Also, the strong axiom $\mathbb{L}.6SS$ will be taken for the input correspondence. The relationship between essentiality of factors and limitationality of output is given by the following several propositions.

<u>Proposition (3.3-1)</u>:

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of n factors is globally input rate weak limitational for a subvector $v^o > 0$ of a vector $u^o = (v^o,w^o)$, $v^o \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$, if it is globally essential for v^o, (axioms $\mathbb{E}.S$ and $\mathbb{L}.6SS$ apply).

<u>Proof</u>:

When $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is globally essential for $v^o > 0$, $\mathbb{L}(v^o,w^o) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty, and by Proposition (3.1-1) CLOSURE $\mathbb{E}(v^o,w^o) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty. Consider $x \in$ CLOSURE $\mathbb{E}(v^o,w^o)$. Since the set $D(\nu_1,\nu_2, \ldots, \nu_k)$ is nonempty and closed, the distance

$$d(x,D(\nu_1,\nu_2, \ldots, \nu_k)) := \text{Inf} \{||x - y|| : y \in D(\nu_1,\nu_2, \ldots, \nu_k)\}$$

is strictly positive and continuous [see (Berge, 1963, p. 84)] in $x \in \text{CLOSURE } \mathbb{E}(v^o,w^o)$. In the norm topology, axiom $\mathbb{E}$.S implies CLOSURE $\mathbb{E}(v^o,w^o)$ is compact, and hence there exists a vector x^* of input rate histories belonging to CLOSURE $\mathbb{E}(v^o,w^o)$ such that

$$0 < d(x^*,D(\nu_1,\nu_2, \ldots, \nu_k)) = \text{Min} \left\{d(x,D(\nu_1,\nu_2, \ldots, \nu_k)) : x \in \text{CLOSURE } \mathbb{E}(v^o,w^o)\right\} .$$

Consider the bound $B(u^o)$ defined by

$$B(u^o) := \frac{d(x^*,D(\nu_1,\nu_2, \ldots, \nu_k))}{2} .$$

Then, since axiom $\mathbb{L}$.6SS applies for the case at hand, $\mathbb{L}(v,w^o) \subset \mathbb{L}(v^o,w^o)$ for any $v \geqq v^o$. Proposition (2.2.4-2) yields $\mathbb{L}(v,w^o) \cap \left\{x \in (L_\infty)^n_+ : ||x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}||\right\} \leqq B(u^o)$ empty for $v \geqq v^o$. Thus $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is globally weak limitational for v^o .

<u>Sub-Proposition (3.3-1)</u>:

Proposition (3.3-1) holds under axiom $\mathbb{E}$ if a weak* topology is used for $(L_\infty)^n_+$.

<u>Proposition (3.3-2)</u>:

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of factors is globally input rate weak limitational for all output rate histories $u > 0$, $u \in (L_\infty)^m_+$, if and only if it is globally essential for $u \in (L_\infty)^m_+$.

Proof:

Assume that $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is not globally essential for u. Then $\mathbb{L}(u^o) \cap D(\nu_1, \nu_2, \ldots, \nu_k) \neq \emptyset$ for some $u^o > 0$ and there exists $x^o \in \mathbb{L}(u^o) \cap D(\nu_1, \nu_2, \ldots, \nu_k)$. By the axiom $\mathbb{L}$.4.2, (Section 2.2) there exists for all $\theta \in (0, +\infty)$ a scalar λ_θ such that $\lambda_\theta x^o \in D(\nu_1, \nu_2, \ldots, \nu_k)$ and $\lambda_\theta x^o \in \mathbb{L}(\theta u^o)$. Thus $\{\nu_1, \nu_2, \ldots, \nu_k\}$ cannot be weak limitational for all $u \in (L_\infty)^m_+$. If $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is globally essential it follows from Proposition (3.3-1) that this subset of factors is weak limitational for all $u \in (L_\infty)^m_+$, by using either $\mathbb{E}$.S or $\mathbb{E}$ with the norm or weak* topology respectively. One merely takes $\ell = m$.

Proposition (3.3-2) is a stronger version of Proposition (3.3-1) which holds when considering the limitationality for all vectors $u \in (L_\infty)^m_+$ of all output rate histories.

The arguments for Proposition (3.3-1) may be carried out in an analogous way to establish the following two propositions, for proper subsets $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of factors which are (i) output history homogeneously essential and (ii) homogeneously essential in conjunction with axioms $\mathbb{L}$.6S and $\mathbb{L}$.6 respectively for disposal of outputs.

Proposition (3.3-3):

A proper subset of factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ *is output rate history homogeneously input rate weak limitational for a subvector* $v^o > 0$ *of a vector* $u^o = (v^o, w^o) \in (L_\infty)^m_+$, $v^o \in (L_\infty)^\ell_+$, $\ell \in \{1, 2, \ldots, m\}$, *if it is output rate history homogeneously essential for* v^o *and axioms* $\mathbb{E}$.S *and* $\mathbb{L}$.6S *apply.*

Proposition (3.3-4):

A proper subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ of factors is homogeneously input rate weak limitational for a subvector $v^o > 0$ of a vector $u^o = (v^o,w^o) \in (L_\infty)^m_+$, $v_o \in (L_\infty)^\ell_+$, $\ell \in \{1,2, \ldots, m\}$, if it is homogeneously essential for v^o .

For the first of these two propositions the fact that $\mathbb{L}\left(\theta_1 v_1^o, \ldots, \theta_\ell v_\ell^o, w^o\right) \subset \mathbb{L}(v^o,w^o)$ for $\theta_i \in [1,+\infty)$, $i \in \{1, \ldots, \ell\}$ is used to show that $\left(\theta_1 v_1^o, \ldots, \theta_\ell v_\ell^o, w^o\right) \notin \mathbb{P}(x)$ for $\theta_i \in [1,+\infty)$, $i \in \{1, \ldots, \ell\}$, when $||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B(u^o)$. For the second proposition, $\mathbb{L}(\theta v^o,w^o) \subset \mathbb{L}(v^o,w^o)$ for $\theta \in [1,+\infty)$ is used.

As in the case of Proposition (3.3-1) there are two sub-propositions related to (3.3-3) and (3.3-4).

Sub-Proposition (3.3-3):

Proposition (3.3-3) holds under axiom $\mathbb{E}$ if a weak* topology is used for $(L_\infty)^n_+$.

Sub-Proposition (3.3-4):

Proposition (3.3-4) holds under axiom $\mathbb{E}$ if a weak* topology is used for $(L_\infty)^n_+$.

Now essentiality will not imply strong limitationality. To see this consider the following counterexample which is a modified version of expression in (2.3-20) (dynamic C.E.S. production function) for two input rate histories $(x_1,x_2) \in X$ (see Section 2.3-5).

$$\mathbb{F}(x_1,x_2) := 0 \in (L_\infty)_+ \quad \text{for} \quad (x_1,x_2) \neq X \ , \ (x_1,x_2) \in (L_\infty)_+^2 \ ,$$

$$\mathbb{F}(x_1,x_2) := u \in (L_\infty)_+ \quad \text{for} \quad (x_1,x_2) \in X \ ,$$

where

$$u(\nu) = F(X_{1\nu},X_{2\nu},\nu) \quad \text{for} \quad t \in [(\nu - 1),\nu) \ , \ \nu = 1,2, \ldots$$

$$F(X_{1\nu},X_{2\nu},\nu) := \begin{cases} \left[\beta_{1\nu}X_{1\nu}^{-\rho} + \beta_{2\nu}X_{2\nu}^{-\rho}\right]^{-\frac{1}{\rho}} & \text{for} \quad (X_{2\nu} - A_\nu) \geqq 0 \\ 0 \quad \text{for} \quad (X_{2\nu} - A_\nu) < 0 \ , \end{cases}$$

and $A_\nu = A$ for $\nu = 1,2, \ldots$ and $\rho \in [-1,0)$. This simplified neoclassical dynamic production function satisfies the same axiom structure as that satisfied by (2.3-20). The second factor for $\mathbb{F}(x_1,x_2)$ is essential, since $\mathbb{F}(x_1,0) = 0 \in (L_\infty)_+$ for $(x_1,0) \in X$. For any bound on X_2 such that $(X_{2\nu} - A_\nu) < 0$ for all $\nu = 1,2, \ldots$, the output rate history generated is bounded, since $F(X_{1\nu},X_{2\nu},\nu) = 0$ for $\nu = 1,2, \ldots$. However for bounds on x_2 such that $(X_{2\nu} - A_\nu) \geqq 0$, $F(X_{1\nu},X_{2\nu},\nu)$ is not bounded for unbounded increase of x_1 .

To serve the investigation of a sufficient condition for strong limitationality, consider $u = (v,w) \in (L_\infty)_+^m$, $v \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$, $\mathbb{L}(v,w) \neq \emptyset$, $v > 0$, with $\{\nu_1,\nu_2, \ldots, \nu_k\}$ as a proper subset of factors essential for the subvector v . It is convenient to use the following definitions:

$$\bar{D}(\nu_1,\nu_2, \ldots, \nu_k) := \left\{ x \in (L_\infty)_+^n : x \neq 0 \ , \ x_{\nu_i} = 0 \ , \ i \in \{1,2, \ldots, k\} \right\} . \tag{3.3-1}$$

$$(3.3\text{-}2)\quad k(\theta v,w) := \left\{x \in (L_\infty)^n_+ : x = \lambda y\ ,\ \lambda \in (0,+\infty)\ ,\ y \in \text{CLOSURE}\ \mathbb{E}(\theta v,w)\right\} .$$

$$(3.3\text{-}3)\qquad K(v,w) := \text{CLOSURE} \bigcup_{\theta\in(0,+\infty)} k(\theta v,w) .$$

$$(3.3\text{-}4)\quad S^o = K(v,w) \cap \left\{x \in (L_\infty)^n_+ : \left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\} .$$

As a preliminary step, the following proposition will be shown:

<u>Proposition (3.3-5)</u>:

If $K(v,w) \cap \bar{D}(\nu_1,\nu_2, \ldots, \nu_k)$ is empty, S^o is closed and bounded for each positive bound $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$, $\mathbb{L}(v,w) \neq \emptyset$.

Let $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ be any positive bound upon the subvector $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)$ of input histories. Since S^o is the intersection of two closed sets it is closed. In order to show that S^o is bounded, assume that $\{x^\alpha\} \subset S^o$ is an infinite sequence with $||x^\alpha|| \to +\infty$ for $\alpha \to \infty$. The subvectors $\left(x^\alpha_{\nu_{k+1}}, x^\alpha_{\nu_{k+2}}, \ldots, x^\alpha_{\nu_n}\right)$ must be such that

$$\left|\left|\left(x^\alpha_{\nu_{k+1}}, x^\alpha_{\nu_{k+2}}, \ldots, x^\alpha_{\nu_n}\right)\right|\right| \to +\infty .$$

Let $\rho^\alpha := \{\lambda x^\alpha : \lambda \in (0,+\infty)\}$. Since the subvectors $\left(x^\alpha_{\nu_1}, x^\alpha_{\nu_2}, \ldots, x^\alpha_{\nu_k}\right)$ are uniformly bounded for $\alpha = 1,2, \ldots$, $\lim\limits_{\alpha\to\infty} \rho^\alpha \in \bar{D}(\nu_1,\nu_2, \ldots, \nu_k)$. Then since $K(v,w)$ is closed, $K(v,w) \cap \bar{D}(\nu_1,\nu_2, \ldots, \nu_k)$ is not empty, a contradiction.

With this proposition as a lemma, one may now express a sufficient condition for globally strong limitational proper subsets of factors $\{\nu_1,\nu_2, \ldots, \nu_k\}$ by the following proposition.

Proposition (3.3-6):

If $K(v,w) \cap \bar{D}(\nu_1,\nu_2, \ldots, \nu_k)$ is empty, $u = (v,w) \in (L_\infty)_+^m$, $\mathbb{L}(v,w) \neq \emptyset$, the subset of factors $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is globally input rate strong limitational for $v \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$ under axiom $\mathbb{L}$.6SS.

Proof:

Let $\left(x^o_{\nu_1},x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ be an arbitrary positive bound for the sub-vector $\left(x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}\right)$ of input histories. Since $K(v,w) \cap \bar{D}(\nu_1,\nu_2, \ldots, \nu_k)$ is empty, it follows from Proposition (3.3-5) that

$$K(v,w) \cap \left\{x \in (L_\infty)_+^n : \left(x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1},x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\}$$

is closed and bounded. Then there exists a positive scalar θ^o such that CLOSURE $\mathbb{E}(\theta v,w) \cap S^o$ is empty for $\theta \geqq \theta^o \geqq 1$, otherwise $\bigcap_{j=1}^{\infty} \mathbb{L}(\theta_j v,w)$ is not empty for $||(\theta_j v,w)|| \to +\infty$ for some infinite sequence $\theta_j \to +\infty$, contradicting axiom $\mathbb{L}$.2, since $\mathbb{L}(\theta v,w) \subset \left(\text{CLOSURE } \mathbb{E}(\theta v,w) + (L_\infty)_+^n\right)$. It then also follows that

$$\mathbb{L}(\theta v,w) \cap \left\{x \in (L_\infty)_+^m : \left(x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1},x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\}$$

is empty for $\theta \geqq \theta^o$. Further since axiom $\mathbb{L}$.6SS applies, define $\bar{v} = \theta^o v \geqq v$, and

$$\mathbb{L}(v,w) \cap \left\{x \in (L_\infty)_+^m : \left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\}$$

for all $v \geqq \bar{v}$, implying $(v,w) \notin \mathbb{P}(x)$ for $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ with the other factors unrestricted. Thus $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is globally strong limitational for $v > 0$.

Similar sufficient conditions for strong limitationality may be developed for output history homogeneously strong limitational and homogeneously strong limitational proper subsets of factors. One need only modify the argument for Proposition (3.3-6) in these two cases to use the axioms $\mathbb{L}.6S$ and $\mathbb{L}.6$ respectively to obtain:

Proposition (3.3-7):

If $K(v,w) \cap \bar{D}(v_1, v_2, \ldots, v_k)$ is empty, $u = (v,w) \in (L_\infty)_+^m$, $v \in (L_\infty)_+^\ell$, $\mathbb{L}(v,w) \neq \emptyset$, the subset of factors is output history homogeneously or homogeneously input rate strong limitational, if axioms $\mathbb{L}.6S$ or $\mathbb{L}.6$ respectively apply.

The efficient subsets $\mathbb{E}(v,w)$ may be taken to satisfy either $\mathbb{E}.S$ or $\mathbb{E}$ by using the norm or weak* topology.

3.4 Limitation of Summable Output Rate Histories by Input Rates of Essential Factors of Production

In the previous section vectors of output rate histories have been taken from the space $(L_\infty)_+^m$, including those which do not correspond to a finite total amount over the time span $t \in [0,+\infty)$. Such total-amount-unbounded planning may seem too expensive. One would expect that most planning would be for a bounded total amount of each kind of output,

probably over a finite subinterval of $[0,+\infty)$. In the treatment of such cases one need not select bounded subintervals of planning, which could hardly be done with generality. Rather the vectors u of output histories may be selected from $(L_\infty)_+^m$ so that each component history is summable.

For this purpose let

$$(3.4\text{-}1) \quad (\tilde{L}_\infty)_+^m := \left\{ u \in (L_\infty)_+^m : \int_0^\infty u_i(t)d\nu_i(t) < +\infty \, , \, i \in \{1,2, \ldots, m\} \right\} .$$

Then output histories which have bounded support (finite planning period) are merely special cases. For vectors $u \in (\tilde{L}_\infty)_+^m$ the supports of the summable component histories u_i need not be bounded.

Correspondingly, there would be little purpose in including for the sets $\mathbb{L}(u)$, $u \in (\tilde{L}_\infty)_+^m$, input vectors x with component histories x_i which are not summable. Thus, define

$$(3.4\text{-}2) \quad (\tilde{L}_\infty)_+^n := \left\{ x \in (L_\infty)_+^n : \int_0^\infty x_i(t)d\mu_i(t) < +\infty \, , \, i \in \{1,2, \ldots, n\} \right\} .$$

Then the sets of vectors of input histories of interest are:

$$(3.4\text{-}3) \quad \tilde{\mathbb{L}}(u) = \left\{ x \in (L_\infty)_+^n : x \in \mathbb{L}(u) \cap (\tilde{L}_\infty)_+^n \right\} , \; u \in (\tilde{L}_\infty)_+^m$$

with efficient subsets:

$$(3.4\text{-}4) \quad \tilde{\mathbb{E}}(u) = \left\{ x \in (L_\infty)_+^n : x \in \tilde{\mathbb{L}}(u) \, , \, y \leq x \Rightarrow y \notin \tilde{\mathbb{L}}(u) \right\} .$$

The definitions previously given, i.e., (3.1-1), (3.1-2), (3.1-3), for $x \varepsilon (L_\infty)^n_+$, $u \varepsilon (L_\infty)^m_+$, may be applied here, since $\tilde{\mathbb{L}}(u) \subset \mathbb{L}(u)$ and $\mathbb{L}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset$ implies $\tilde{\mathbb{L}}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ empty. A subset $\{\nu_1,\nu_2, \ldots, \nu_k\}$ which is essential in the sense of any of the three types previously defined is likewise essential when $u \varepsilon (\tilde{L}_\infty)^m_+$, $x \varepsilon (\tilde{L}_\infty)^n_+$.

Propositions (3.3-1), (3.3-3), (3.3-4) and Sub-Propositions (3.3-1), (3.3-3), (3.3-4) carry over when $u = (v,w)$ is restricted to $(\tilde{L}_\infty)^m_+$. It suffices to consider Proposition (3.3-1) for verification of these facts. The global essentiality of the subsets $\{\nu_1,\nu_2, \ldots, \nu_k\}$ for $v^o > 0$, $u^o = (v^o,w^o) \varepsilon (\tilde{L}_\infty)^m_+$, $v^o \varepsilon (\tilde{L}_\infty)^\ell_+$, $\ell \varepsilon \{1,2, \ldots, m\}$, implies that

$$\text{CLOSURE } \mathbb{E}(v^o,w^o) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset ,$$

as before. Since CLOSURE $\tilde{\mathbb{E}}(v^o,w^o) \subset$ CLOSURE $\mathbb{E}(v^o,w^o)$, it follows that

$$\text{CLOSURE } \tilde{\mathbb{E}}(v^o,w^o) \cap D(\nu_1,\nu_2, \ldots, \nu_k) = \emptyset .$$

Then using CLOSURE $\tilde{\mathbb{E}}(v^o,w^o)$ in place of CLOSURE $\mathbb{E}(v^o,w^o)$, the previous argument for Proposition (3.3-1) may be carried out to obtain a positive bound $\tilde{B}(u^o)$ and show that

$$\tilde{\mathbb{L}}(v,w^o) \cap \left\{x \varepsilon (L_\infty)^m_+ : \left|\left|\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)\right|\right| \leqq \tilde{B}(u^o)\right\}$$

is empty for $v \geqq v^o$, $v \varepsilon (\tilde{L}_\infty)^\ell_+$. For these arguments weak or strong axioms $\tilde{\mathbb{E}}$, $\tilde{\mathbb{E}}$.S may be used for compactness of CLOSURE $\tilde{\mathbb{E}}(u)$ under a weak* or norm topology for $(L_\infty)^n_+$, respectively. The advantages of considering

only summable output histories is that the axioms $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{E}}.S$ are weaker than the axioms $\mathbb{E}$ and $\mathbb{E}.S$ used for the case where histories are not restricted to be summable. Also, only a finite argument is required to show that $\tilde{\mathbb{E}}(u)$ is not empty when u has summable components.

In a similar way Propositions (3.3-6) and (3.3-7) carry over for the situation where output and input vectors are restricted to have summable output rate histories for the definitions of $\tilde{\mathbb{L}}(v,w)$ and $\tilde{\mathbb{E}}(v,w)$. The set $\tilde{S}_o$ so defined is closed since CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o)$ is involved, and bounded since CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o) \subset$ CLOSURE $\mathbb{E}(\theta v^o,w^o)$. Further as in the previous case there exists $\theta^o > 1$ such that CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o) \cap S^o$ is empty for $\theta \geqq \theta^o$ and since CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o) \subset$ CLOSURE $\mathbb{E}(\theta v^o,w^o)$ for $\theta > 0$, it follows that CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o) \cap S^o$ is empty for $\theta \geqq \theta^o$, implying that $\tilde{\mathbb{L}}(\theta v^o,w^o) \cap \left\{x \varepsilon (L_\infty)^n_+ : \left(x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1},x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\}$ is empty for $\theta \geqq \theta^o$, because $\tilde{\mathbb{L}}(\theta v^o,w^o) \subset$ CLOSURE $\tilde{\mathbb{E}}(\theta v^o,w^o) + (L_\infty)^n_+$.

Here one may also use the weaker axioms $\tilde{\mathbb{E}}.S$ and $\tilde{\mathbb{E}}$ with appropriate topology as needed.

To approach this topic from another viewpoint, the original production correspondence may be altered. Since only summable input and output rate histories are of interest for the foregoing discussion, one might ask why not restrict $x \varepsilon (L_1)^n_+$ and $u \varepsilon (L_1)^m_+$. In effect, why not define the production structure as a correspondence

$$x \varepsilon (L_1)^n_+ \to \mathbb{P}_1(x) \varepsilon 2^{(L_1)^m_+} \tag{3.4-5}$$

mapping vectors of summable input rate histories into subsets of vectors of summable output rate histories. But then the norms for $(L_1)^n_+$ and $(L_1)^m_+$ are changed to:

$$(3.4\text{-}6) \quad ||x|| = \operatorname*{Max}_i ||x_i|| \; ; \; ||x_i|| = \int_0^\infty |x_i(t)|d\mu_i(t) \; , \; i \in \{1, \ldots, n\}$$

$$(3.4\text{-}7) \quad ||u|| = \operatorname*{Max}_i ||u_i|| \; ; \; ||u_i|| = \int_0^\infty |u_i(t)|d\nu_i(t) \; , \; i \in \{1, \ldots, m\} \; ,$$

and measure by components the total amount of the good or service involved. With this approach the laws of return are expressed in total amounts of some good or service involved.

The axiom structure of Chapter 2 carries over naturally for the norm topologies of $(L_1)^n_+$, $(L_1)^m_+$. Only those axioms involving the norm need to be reinterpreted. $\mathbb{P}$.1 has the same meaning. Each vector $u \in \mathbb{P}(x)$ of output histories has bounded total amount for each output component. By itself this does not mean that all norms $||u||$ for $u \in \mathbb{P}_1(x)$ are bounded, and axioms $\mathbb{P}$.2, $\mathbb{P}$.2S assure boundedness and total boundedness respectively for the set $\mathbb{P}_1(x)$, implying that for all $u \in \mathbb{P}_1(x)$ each output is bounded in total amount. There is no difference in the interpretation of properties $\mathbb{P}$.3, $\mathbb{P}$.3S and $\mathbb{P}$.3SS for $x \in (L_1)^n_+$, $u \in (L_1)^m_+$ as opposed to the case where $x \in (L_\infty)^n_+$, $u \in (L_\infty)^n_+$. Likewise there is no difference in the interpretation of the properties $\mathbb{P}$.4.1, $\mathbb{P}$.4.2, $\mathbb{P}$.6, $\mathbb{P}$.6S, $\mathbb{P}$.6SS. However for property $\mathbb{P}$.5 convergence is only defined by neighborhoods of the norm topologies for $(L_1)^m_+$, $(L_1)^n_+$.

Concerning the axioms on initiation and time extension of outputs, i.e., $\mathbb{P}.T.1$ and $\mathbb{P}.T.2$, there is no alteration of meaning by going to the spaces $(L_1)^m_+$, $(L_1)^n_+$. It is worth noting here that summability of input or output rate history does not imply boundedness of the supports of these functions.

For the inverse correspondence

$$(3.4\text{-}8) \qquad u \in (L_1)^m_+ \to \mathbb{L}_1(u) \in 2^{(L_1)^n_+} ,$$

the interpretation of properties $\mathbb{L}.1$, $\mathbb{L}.3$, $\mathbb{L}.3S$, $\mathbb{L}.3SS$, $\mathbb{L}.4.1$, $\mathbb{L}.4.2$, $\mathbb{L}.6$, $\mathbb{L}.6S$ and $\mathbb{L}.6SS$ is not altered. However, in property $\mathbb{L}.2$, $||u^\alpha||$ is interpreted as $\underset{i}{\text{Max}}$ {Total amount of the i^{th} output} , and $\{||u^\alpha||\} \to +\infty$ means the sequence is unbounded in the *Maximal Total* Amount for an output rate history. In property $\mathbb{L}.2S$ and $\mathbb{L}.5$, convergence is defined by the neighborhoods for the norm topologies for $(L_1)^m_+$ and $(L_1)^n_+$.

The axioms on time extension of inputs, i.e., $\mathbb{L}.T.1$ and $\mathbb{L}.T.2$ are not altered as to meaning by restriction to the spaces $(L_1)^m_+$, $(L_1)^n_+$.

In the definition of the efficient subset $\mathbb{E}_1(u)$ of $\mathbb{L}_1(u)$ no change is required, since the norm is not used. However, the axioms $\mathbb{E}_1$, $\mathbb{E}_1.S$, i.e., boundedness and total boundedness of the efficient subset, have different meaning, just as in the case of property $\mathbb{P}.2$, $\mathbb{P}.2S$, for $\mathbb{P}_1(x)$. If $\mathbb{E}_1(u)$ is bounded, the maximal total amount of an input for all $x \in \mathbb{E}_1(u)$ is bounded, as opposed to input rates being bounded for $\mathbb{E}(u)$ under axiom $\mathbb{E}$.

With the foregoing changes of meaning for certain parts of the axiom structure for production correspondences, essentiality and limitationality of outputs can be considered.

The definitions (3.1-1), (3.1-2), (3.1-3) for essentiality of a proper subset $\{\nu_1, \nu_2, \ldots, \nu_k\}$ of n factors of production still apply for the correspondences (3.4-5), (3.4-6). All one need to do is to replace $(L_\infty)^m_+$, $(L_\infty)^\ell_+$ by $(L_1)^m_+$ and $(L_1)^\ell_+$.

By replacing $\mathbb{E}.S$ by $\mathbb{E}_1.S$, as well, Proposition (3.3-1) carries over by symmetry. Similarly Sub-Proposition (3.3-1) holds. Likewise Proposition (3.3-2), (3.3-3), (3.3-4), Sub-Propositions (3.3-3) and (3.3-4), and Proposition (3.3-5), (3.3-6), (3.3-7) follow by symmetry.

Thus, provided one interprets the boundary of a component of an output vector or input vector to mean bounding the total amount of that input or output over $[0,+\infty)$ as opposed to bounding input or output rates, the related laws of return are of the same form.

3.5 Limitation of Output Rates by Intervals of Time Over Which Essential Factors are Applied

Here one is concerned with the limitations upon output rates imposed when essential factors of production have limited supports, i.e., limited periods of time with positive rate of application. Only the case of entire vectors $u \in (L_\infty)^m_+$ will be considered.

For the analysis to follow certain definitions are useful:

(3.5-1) $$S(u^o) := \left\{ t \in R^n_+ : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \ldots, \bar{t}_{x_n}\right), x \in \text{CLOSURE}\, \mathbb{E}(u^o) \right\}.$$

(3.5-2) $$S(D(\nu_1, \nu_2, \ldots, \nu_k)) := \left\{ t \in R^n_+ : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \ldots, \bar{t}_{x_k}\right), x \in D(\nu_1, \nu_2, \ldots, \nu_k) \right\}.$$

Definition (3.5-1):

A proper subset of n factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is globally input support weak limitational for an output vector $u^o \in (L_\infty)^m_+$ if there exists a positive bound $T(u^o)$ such that $u \notin \mathbb{P}(x)$ for $u \geqq u^o$ when $\operatorname{Max}\left\{\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\right\} \leqq T(u^o)$ while $\bar{t}_{x_{\nu_j}}$ is unrestricted for $j \in \{(k+1), (k+2), \ldots, n\}$.

Definition (3.5-2):

A proper subset of factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is globally input support strong limitational for output vectors $u \in (L_\infty)^m_+$ if for each positive time bound $T \in R_{++}$ there exists an output vector $u(T) \in (L_\infty)^m_+$ such that $u \notin \mathbb{P}(x)$ for $u \geqq u(T)$ when $\operatorname{Max}\left\{\bar{t}_{\nu_1}, \bar{t}_{\nu_2}, \ldots, \bar{t}_{\nu_k}\right\} \leqq T$.

Then the following proposition characterizes a limitation upon vectors due to limitations on the span of positive input rates for essential factors of production.

Proposition (3.5-1):

If $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is a proper subset of factors globally essential for $u^o \in (L_\infty)^m_+$, $u^o \geq 0$, $\mathbb{L}(u^o) \neq \emptyset$, and $\operatorname{Max}_i \left\{\bar{t}_{u_i^o}\right\} < +\infty$, it is Globally Input Support Weak Limitational for u^o if axiom $\mathbb{L}$.6SS applies.

Note first that $S(u^o)$ is bounded, since Supp u^o is bounded by virtue of axiom $\mathbb{L}$.T.2. Next it is to be shown that

$$(3.5\text{-}3) \qquad (\text{CLOSURE } S(u^o)) \cap S(D(\nu_1, \nu_2, \ldots, \nu_k))$$

is empty. Note that CLOSURE $S(u^o)$ is compact since it is a bounded and closed subset of R^n_+. Assume that (3.5-3) is not empty. Then there exists a vector $\hat{t} \in R^n_+$ such that $\hat{t} \in$ CLOSURE $S(u^o)$ and $\hat{t} \in S(D(\nu_1,\nu_2, \ldots, \nu_k))$, and there exists an infinite sequence $\{t^\alpha\} \subset$ CLOSURE $S(u^o)$ with $t^\alpha \to \hat{t}$. Corresponding to $\{t^\alpha\}$ there exists an infinite sequence $\{x^\alpha\} \subset$ CLOSURE $\mathbb{E}(u^o)$. Now, either by axiom $\mathbb{E}$.S with norm topology for $(L_\infty)^n_+$ or by axiom $\mathbb{E}$ with weak* topology for $(L_\infty)^n_+$, CLOSURE $\mathbb{E}(u^o)$ is compact. Then there exists a subsequence $\left\{x^{\alpha_k}\right\} \subset$ CLOSURE $\mathbb{E}(u^o)$ such that $x^{\alpha_k} \to x^o$ and $x^o \in$ CLOSURE $\mathbb{E}(u^o)$. Further let $t^o \in S(u^o)$ for x^o . Now $t^o = \hat{t}$, otherwise $x^{\alpha_k} \not\to x^o$. Since $x^o \notin D(\nu_1,\nu_2, \ldots, \nu_k)$ due to the essentiality of $\{\nu_1,\nu_2, \ldots, \nu_k\}$ for u^o , it follows that $t^o \notin S(D(\nu_1,\nu_2, \ldots, \nu_k))$, a contradiction.

Now since CLOSURE $S(u^o)$ is compact and $S(D(\nu_1,\nu_2, \ldots, \nu_k))$ is a closed set, the distance, defined for $t \in$ CLOSURE $S(u^o)$, and given by

$$\Delta(t,S(D(\nu_1,\nu_2, \ldots, \nu_k)) := \operatorname*{Inf}_{\sigma} \{||\sigma - t|| : \sigma \in S(D(\nu_1,\nu_2, \ldots, \nu_k))\} ,$$

is continuous in t , and strictly positive since CLOSURE $S(u^o) \cap S(D(\nu_1,\nu_2, \ldots, \nu_k))$ is empty. Then, since CLOSURE $S(u^o)$ is a nonempty compact set, there exists an input vector with $t^* \in$ CLOSURE $S(u^o)$ such that

$$0 < \delta := \operatorname*{Min}_{t} \{\Delta(t,S(D(\nu_1,\nu_2, \ldots, \nu_k)) : t \in \text{CLOSURE } S(u^o)\} .$$

Let $T(u^o) := \delta/2$ be a bound upon the norm of the subvector $\left(\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\right)$. Define $\bar{t}_x = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \ldots, \bar{t}_{x_n}\right)$. Then for $\bar{t}_x \in \left\{t \in R^n_+ : \|\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\| \leqq T(u^o)\right\}$, CLOSURE $\mathbb{E}(u^o) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty, since (CLOSURE $S(u^o)) \cap S(D(\nu_1,\nu_2, \ldots, \nu_k))$ is empty and input vectors x with $\bar{t}_x$ so bounded cannot belong to those of CLOSURE $\mathbb{E}(u^o)$. Further, since, $\mathbb{L}(u^o) \subset \left(\text{CLOSURE } \mathbb{E}(u^o) + (L_\infty)^n_+\right)$, see Proposition (2.2.4-2), it follows that

$$\left\{t \in R^n_+ : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \ldots, \bar{t}_{x_n}\right), x \in \mathbb{L}(u^o)\right\} \subset \text{CLOSURE}\left(S(u^o) + R^n_+\right).$$

Moreover, if axiom $\mathbb{L}$.6SS applies, $\mathbb{L}(u) \subset \mathbb{L}(u^o)$ for $u \geqq u^o$. Thus, it *is shown that*, if $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is an essential proper subset of factors for an output vector $u^o \in (L_\infty)^m_+$, there exists a positive bound $T(u^o)$ such that $u \notin \mathbb{P}(x)$ for $\underset{i}{\text{Max}}\left\{\bar{t}_{x_{\nu_i}}\right\} > T(u^o)$ and $u \geqq u^o$.

Essentiality of a proper subset of factors therefore engenders limitation upon output rate histories relative to a given output vector u^o when output histories of essential factors for u^o have supports not exceeding a bound $T(u^o)$ depending upon u^o and $\bar{t}^o_{u_i}$ is bounded for all $i \in \{1,2, \ldots, m\}$. For each vector $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, of output histories with bounded supports, output rate histories may be so bounded by restrictions on the supports of factors essential for u. For any given bound $T(u^o)$ not all output vectors $u \in (L_\infty)^m_+$ need by the axioms have bounded supports of output rate histories. Only when Supp $u_i \supset$ Supp u^o_i, $i \in \{1,2, \ldots, m\}$, *and* $u \geqq u^o$, does $\text{Max}\left\{\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\right\} \leqq T(u^o)$ exclude such output vectors u.

Even in the case $m = 1$ and $u^o \varepsilon (L_\infty)_+$, the same issue arises, i.e., $v \varepsilon (L_\infty)_+$ need not be restricted in support when $v \not\geq u^o$.

Following the treatment given earlier for limitation by input rates when $\mathbb{L}$.6S and $\mathbb{L}$.6 apply, one may define

Definition (3.5-3):

A proper subset of factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is output rate history homogeneously input support weak limitational for an output vector $u^o \varepsilon (L_\infty)^m_+$ if there exists a positive bound $T(u^o)$ such that $\left(\theta_1 u^o_1, \ldots, \theta_m u^o_m\right) \notin \mathbb{P}(x)$ for $\theta_i \varepsilon [1,+\infty)$, $i \varepsilon \{1,2, \ldots, m\}$ when $\text{Max}\left\{\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\right\} \leqq T(u^o)$ while $\bar{t}_{x_{\nu_j}}$ is unrestricted for $j \varepsilon \{(k+1), (k+2), \ldots, n\}$.

Definition (3.5-4):

A proper subset of factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is homogeneously input support weak limitational for an output vector $u^o \varepsilon (L_\infty)^m_+$ if there exists a positive bound $T(u^o)$ such that $(\theta u^o) \notin \mathbb{P}(x)$ for $\theta \varepsilon [1,+\infty)$ when $\text{Max}\left\{\bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \ldots, \bar{t}_{x_{\nu_k}}\right\} \leqq T(u^o)$ while $\bar{t}_{x_{\nu_j}}$ is unrestricted for $j \varepsilon \{(k+1), (k+2), \ldots, n\}$.

Then the following proposition is a consequence of the arguments of Proposition (3.5-1):

Proposition (3.5-2):

If $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is a proper subset of factors essential for $u^o \varepsilon (L_\infty)^m_+$, $u^o \geq 0$, $\mathbb{L}(u^o) \neq \emptyset$, and $\underset{i}{\text{Max}}\left\{\bar{t}_{u_i}\right\} < +\infty$, it is homogeneously input support weak limitational for u^o if axiom $\mathbb{L}$.6 applies.

Here, by the very scaling of output histories permitted by the axioms $\mathbb{L}.6S$ and $\mathbb{L}.6$ for disposal of outputs, the supports of the output vectors so scaled and compared are identical, and the supports of the output vectors $\left(\theta_1 u_1^o, \ldots, \theta_m u_m^o\right)$ and θu^o, respectively are trivially bounded for $\theta_i \in [1,+\infty)$, $i \in \{1,2, \ldots, m\}$ and $\theta \in [1,+\infty)$.

3.6 Equivalent Specifications of Strong Limitationality

A characterization of strong limitationality equivalent to that given by Definition (3.3-2) may be made in terms of the efficient subset $\mathbb{E}(v,w)$ of an output vector $u = (v,w)$, $v \in (L_\infty)_+^\ell$, $\ell \in \{1,2, \ldots, m\}$, $u \in (L_\infty)_+^m$.

Proposition (3.6-1):

Global Strong Limitationality of $\{\nu_1,\nu_2, \ldots, \nu_k\}$ for a subvector $v \in (L_\infty)_+^\ell$ of $u = (v,w) \in (L_\infty)_+^m$, $\ell \in \{1,2, \ldots, m\}$, is equivalent to: For each positive bound $B \in R_{++}$ there exists a subvector $v(B) \in (L_\infty)_+^\ell$, $v(B) > 0$ such that $(\text{CLOSURE}\, \mathbb{E}(v,w)) \cap x \in \left\{(L_\infty)_+^n : \|x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\| \leqq B\right\}$ is empty for $v \geqq v(B)$, $u = (v,w)$, when $\mathbb{L}.6SS$ holds.

If the intersection stated is empty for $v \geqq v(B)$, then by $\mathbb{L}(v(B),w) \subset \left(\text{CLOSURE}\ \mathbb{E}(v(B),w) + (L_\infty)_+^n\right)$, see Proposition (2.2-4-2), and by $\mathbb{L}.6SS$, $\mathbb{L}(v,w) \subset \mathbb{L}(v(B),w)$ for $v \geqq V(B)$, it follows that $\mathbb{L}(v,w) \cap \left\{x \in (L_\infty)_+^n : \|x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\| \leqq B\right\}$ is empty for $v \geqq v(B)$. Accordingly $(v,w) \notin \mathbb{P}(x)$ for $v \geqq v(B)$ when $\|x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\| \leqq B$, and $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is strong limitational for subvectors $v \in (L_\infty)_+^\ell$.

Next, assume that $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is globally input rate strong limitational for the subvector $v \in (L_\infty)^\ell_+$. Then for each positive bound B there is a subvector $v(B) \in (L_\infty)^\ell_+$ such that $\mathbb{L}(v,w) \cap \left\{x \in (L_\infty)^n_+ : ||x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}$ is empty for all $v \geqq v(B)$. Since $\mathbb{E}(v,w) \subset \mathbb{L}(v,w)$, and $\mathbb{L}(u)$ is closed, $(\text{CLOSURE } \mathbb{E}(v,w)) \cap \left\{x \in (L_\infty)^n_+ : ||x_{\nu_1},x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}$ is empty for $v \geqq v(B)$ and the intersection stated is empty.

Economically, this proposition means that if a proper subset of factors $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is essential for a subvector $v \in (L_\infty)^\ell_+$ of $u = (v,w) \in (L_\infty)^m_+$, $\ell \in \{1,2, \ldots, m\}$, and the input rates of these essential inputs are bounded, there exists an output subvector $v(B)$ such that increases of the time histories of $v(B)$ are not possible without increasing the input rates of the essential factors.

Similarly, in terms of the limitation of output rate histories by bounds upon the supports of essential factors.

<u>Proposition (3.6-2)</u>:

Global Strong Limitationality of $\{\nu_1,\nu_2, \ldots, \nu_k\}$ for vectors $u \in (L_\infty)^m_+$ is equivalent to: For each positive time bound $T \in R_{++}$ there exists an output vector $u(T) \in (L_\infty)^m_+$, $u(T) > 0$ such that $S(u) \cap S(D(\nu_1,\nu_2, \ldots, \nu_k))$ is empty for $u \geqq u(T)$.

If $D(u) \cap S(D(\nu_1,\nu_2, \ldots, \nu_k))$ is empty for $u \geqq u(T)$, then by the arguments of Section 3.5,

$$(3.6\text{-}1) \qquad \mathbb{L}(u) \cap \left\{x \in (L_\infty)^n_+ : \text{Max}\left\{\bar{t}_{\nu_1},\bar{t}_{\nu_2}, \ldots, \bar{t}_{\nu_k}\right\} \leqq T\right\}$$

is empty for $u \geqq u(T)$, and $u \notin \mathbb{P}(x)$ for $u \geqq u(T)$ when $\max\left\{\bar{t}_{\nu_1}, \bar{t}_{\nu_2}, \ldots, \bar{t}_{\nu_k}\right\} \leqq T$.

On the other hand, if $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is globally input support strong limitational for output vector $u \in (L_\infty)_+^m$, then for each positive time bound $T \in R_{++}$, the intersection (3.6-1) is empty for $u \geqq u(T) > 0$, implying

$$\text{CLOSURE } \mathbb{E}(u) \cap \left\{x \in (L_\infty)_+^n : \text{Max}\left\{\bar{t}_{\nu_1}, \bar{t}_{\nu_2}, \ldots, t_{\nu_k}\right\} \leqq T\right\}$$

is empty for $u \geqq u(T)$, and

$$S(u) \cap S(D(\nu_1, \nu_2, \ldots, \nu_k))$$

is empty for $u \geqq u(T)$. Thus the proposition holds.

Here the economic meaning of the equivalent statement for strong limitationality by essential factor supports is one where increases of time histories of $u(T)$ are not possible without increasing the supports (periods of time of positively measurable input rate) of the essential factors.

There is nothing in the general properties postulated by the axioms which implies that essential factors be strong limitational. However, as more special structures are considered, the intuitively appealing strong limitationality of essential factors may emerge. Compare Proposition (3.3-7) with Definition (5.1-2) of chapter 5 below.

3.7 Restriction of Output by Nondisposability of Factors

Linking of factors in production may result in restraints upon output when input of a factor link is restricted. Similar to the jointness of outputs (see Section 3.2), null jointness of factors is defined by:

Definition (3.7-1): (Färe and Jansson, 1976)

A proper subset $\{\nu_{k+1}, \nu_{k+2}, \ldots, \nu_n\}$ of n factors is Null Joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$ for $u > 0$, $\mathbb{L}(u) \neq \emptyset$, if $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) = 0$ implies $\left(x_{\nu_{k+1}}, \ldots, x_{\nu_n}\right) = 0$ when $x \in C(u) := \text{CLOSURE}\left\{x \in (L_\infty)_+^n : x = \lambda y,\ \lambda > 0,\ y \in \text{ISOQ}\ \mathbb{L}(u)\right\}$.

Note that by the axiom $\mathbb{L}.4.2$, $C(\theta u) = C(u)$ for $\theta \in (0, +\infty)$. Another way of specifying Null Jointness of factors is given by the following proposition.

Proposition (3.7-1):

A proper subset $\{\nu_{k+1}, \ldots, \nu_n\}$ of n factors is null joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$ for $u > 0$, $\mathbb{L}(u) \neq \emptyset$, if and only if $C(u) \cap \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$ is empty.

First assume that there exists $x \in C(u) \cap \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$. Then for this vector of input histories, $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) = 0$ with $\left(x_{\nu_{k+1}}, \ldots, x_{\nu_n}\right) \neq 0$, since $x \geq 0$ because $x \in C(u)$ implies $x \neq 0$ for $u > 0$, $\mathbb{L}(u) \neq \emptyset$. Thus if $C(u) \cap \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$ is not empty, $\{\nu_{k+1}, \ldots, \nu_n\}$ is not null joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$. Conversely, assume that $\{\nu_{k+1}, \ldots, \nu_n\}$ is not null joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$.

Then there exists $x \varepsilon C(u)$ with $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) = 0$ and $\left(x_{\nu_{k+1}}, \ldots, x_{\nu_n}\right) \geq 0$. Accordingly, $x \varepsilon \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$ and $C(u) \cap \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$ is not empty.

Note that, as defined in Section (3.3), $K(u) := \text{CLOSURE} \bigcup_{\theta>0} K(\theta u)$ is a subset of $C(u)$. Hence, by Proposition (3.3-6), (3.3-7), we have

Proposition (3.7-2):

If the proper subset $\{\nu_{k+1}, \ldots, \nu_n\}$ is Null Joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$ for $u > 0$, $\mathbb{L}(u) \neq \emptyset$, the subset of factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is input rate strong limitational for the production of u.

Depending upon whether axiom $\mathbb{L}$.6SS, or $\mathbb{L}$.6S or $\mathbb{L}$.6 applies, the strong limitationality of Proposition (3.7-2) is global, input history homogeneous or merely homogeneously limitational.

A proper subset of n factors $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is regarded as Congested by $\{\nu_{k+1}, \ldots, \nu_n\}$ in the production of $u > 0$, $\mathbb{L}(u) \neq \emptyset$, if bounds upon the input rate histories $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)$ requires that the input rate histories $\left(x_{\nu_{k+1}}, \ldots, x_{\nu_n}\right)$ are bounded. The following definition makes this notion precise.

Definition (3.7-2):

The proper subset of n factors $\{\nu_{k+1}, \ldots, \nu_n\}$ is CONGESTED in the production of $u > 0$, $\mathbb{L}(u) \neq \emptyset$, by $\{\nu_1, \nu_2, \ldots, \nu_k\}$ if for each positive bound $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ on $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)$ there is a positive number $N \varepsilon R_{++}$ such that

$$\left\{x \,\varepsilon\, (L_\infty)^n_+ : x \,\varepsilon\, \mathbb{L}(\theta u)\,,\ \left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right), \ ||x|| > N\right\}$$

is empty for all $\theta \,\varepsilon\, (0,+\infty)$.

Now by an argument paralleling that given for Proposition (3.3-5), the following lemma holds:

Lemma:

Let $K \subset (L_\infty)^n_+$, $K \neq \{0\}$, $0 \,\varepsilon\, K$ be a closed cone such that $K \cap \bar{D}(\nu_1, \nu_2, \ldots, \nu_k)$ is empty $(1 \leqq k < n)$. Then the intersection set

$$K \cap \left\{x \,\varepsilon\, (L_\infty)^n_+ : \left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)\right\}$$

is closed and bounded for each positive bound $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ on the subvector $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)$.

The significance of Null Jointness of inputs for congestion of output is given by:

Proposition (3.7-3):

If the subset of n factors $\{\nu_{k+1}, \ldots, \nu_n\}$ is null joint with $\{\nu_1, \nu_2, \ldots, \nu_k\}$ *in the production of* $u > 0$, $\mathbb{L}(u) \neq 0$, $\{\nu_{k+1}, \ldots, \nu_n\}$ is congested by $\{\nu_1, \nu_2, \ldots, \nu_k\}$.

This proposition follows directly from Proposition (3.7-1) and the lemma. It is from such a proposition that so-called laws of variable return may be deduced by considering production correspondences $u \to \mathbb{L}(u)$ with further assumption on fine structure. Proposition (3.7-3)

states in a general way that null joint factors are linked so that beyond a certain size (norm) of input vector all output vectors of a given mix (for which the factors are null joint) are impossible. How, under increases of input rates, an output vector of the form (θu) may increase and decrease to zero, is a matter of fine structure not considered here.

CHAPTER 4

FUNCTIONAL REPRESENTATION OF DYNAMIC PRODUCTION CORRESPONDENCES

The mapping (correspondences) of Chapter 2 serve well to define the dynamic structure of production. Even so, it is useful for several reasons to obtain a representation of production alternatives in terms of functionals.

4.1 The Output Distance Functional

For $(x,u) \in (L_\infty)_+^n \times (L_\infty)_+^m$, the output distance functional $\Omega : (L_\infty)_+^n \times (L_\infty)_+^m \to R_+$ is defined (Shephard, 1970:a) by

$$(4.1\text{-}1) \qquad \Omega(x,u) := [\mathrm{Max}\ \{\theta \in R_+ : (\theta u) \in \mathbb{P}(x)\}]^{-1} .$$

In order to see that this functional provides a means of defining the correspondence $x \to \mathbb{P}(x)$, consider the cases $\mathbb{P}(x) = \{0\}$, $\mathbb{P}(x) \neq \{0\}$, for $u = 0$ and $u \geq 0$. When $\mathbb{P}(x) = \{0\}$ and $u \geq 0$, $\Omega(x,u) = +\infty$, while $\Omega(x,0) = 0$ for $u = 0$. When $\mathbb{P}(x) \neq \{0\}$ and $u = 0$, $\Omega(x,0) = 0$ as in the previous case. However, when $u \geq 0$ there are two situations to consider

(i) $\{\theta u : \theta \in [0,+\infty)\} \cap \mathbb{P}(x) \neq \{0\}$

(ii) $\{\theta u : \theta \in [0,+\infty)\} \cap \mathbb{P}(x) = \{0\}$.

In case (i), clearly

$$0 < \Omega(x,u) < \infty$$

and if $u \in \mathbb{P}(x)$

$$\Omega(x,u) \leqq 1 \ .$$

In case (ii), $\Omega(x,u) = +\infty$. By the cases outlined, if $\Omega(x,u) \leqq 1$, then $u \in \mathbb{P}(x)$. Hence the following proposition holds:

Proposition (4.1-1): (Shephard, 1970:a)

$u \in \mathbb{P}(x)$ if and only if $\Omega(x,u) \leqq 1$, $x \in (L_\infty)^n_+$, i.e., $\mathbb{P}(x) = \left\{u \in (L_\infty)^n_+ : \Omega(x,u) \leqq 1\right\}$.

With isoquant of $\mathbb{P}(x)$ defined by

$$\text{(4.1-2)} \quad \text{ISOQ}\,\mathbb{P}(x) := \begin{cases} \{u \in \mathbb{P}(x) : (\theta u) \notin \mathbb{P}(x) \ , \ \theta \in (1,+\infty)\} \ , & \mathbb{P}(x) \neq \{0\} \\ \{0\} \ , & \mathbb{P}(x) = \{0\} \ . \end{cases}$$

The following proposition also holds.

Proposition (4.1-2): (Shephard, 1970:a)

$u \in \text{ISOQ}\ \mathbb{P}(x)$ if and only if

$$\Omega(x,u) = 1 \ , \ \mathbb{P}(x) \neq \{0\} \ .$$

As defined by the expressions (4.1-1), the isoquant of an output set $\mathbb{P}(x)$ is the dynamic generalization of the familiar Production Frontier. In this context, the production frontier is composed of those vectors of output rate histories realizable by x which cannot be homogeneously scaled upward for x , preserving the time distributions of output rates. All such vectors u of $\mathbb{P}(x)$ are defined by the functional equation

$$\text{(4.1-3)} \qquad \Omega(x,u) = 1 \ .$$

For this purpose it behooves us to investigate the properties of the functional $\Omega(x,u)$ as they are implied by those of the map sets $\mathbb{P}(x)$. Without detailed argument they are stated in the following proposition.

<u>Proposition (4.1-3)</u>: (Shephard, 1970:a)

The distance functional $\Omega(x,u)$ satisfies:

Ω.1 $\Omega(x,u) = +\infty$ for $\mathbb{P}(x) = \{0\}$, $u \geq 0$, and $\Omega(x,0) = 0$ for $x \in (L_\infty)^n_+$.

Ω.2 $\Omega(x,u) \geqq 0$, $x \in (L_\infty)^n_+$, $u \in (L_\infty)^m_+$, and $\Omega(x,u) \geq 0$ for $u \geq 0$.

Ω.3 $\Omega(x,\theta u) = \theta\Omega(x,u)$, $\theta \in (0,+\infty)$, $x \in (L_\infty)^n_+$, $u \in (L_\infty)^m_+$.

Ω.4 $\Omega(\lambda x,u) \leqq \Omega(x,u)$, $\lambda \in [1,+\infty)$, $x \in (L_\infty)^n_+$, $u \in (L_\infty)^m_+$.

Ω.5 $\Omega(x,u)$ is lower semi-continuous in $x \in (L_\infty)^n_+$ and in $u \in (L_\infty)^m_+$.

The first three properties are obvious from definitions, and property Ω.4 is a direct result of property $\mathbb{P}$.3 for $\mathbb{P}(x)$. From the stronger forms $\mathbb{P}$.3S and $\mathbb{P}$.3SS, one obtains

Ω.3S $\Omega(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, u) \leqq \Omega(x,u)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$.

Ω.3SS $\Omega(x',u) \leqq \Omega(x,u)$ for $x' \geqq x$.

Property Ω.5 may be verified as follows:

<u>Definition (4.1-1)</u>:

$\Omega(x,u)$ is lower semi-continuous in $x \in (L_\infty)^n_+$ for $u \in (L_\infty)^m_+$, if and only if the set $\{x : \Omega(x,u) \leqq \alpha\}$ is closed for all real numbers α .

Clearly the set $\{x : \Omega(x,u) \leqq \alpha\}$ is empty and closed for $\alpha < 0$. Concerning $\{x : \Omega(x,u) = 0\}$, (i), $\Omega(x,0) = 0$ for all $x \in (L_\infty)^n_+$ and $\{x : \Omega(x,0) = 0\} = (L_\infty)^n_+$, closed. In case (ii), $\Omega(x,u) \neq 0$ for all $x \in (L_\infty)^n_+$, and $\{x : \Omega(x,u) = 0\}$ is empty and closed. It remains only to consider $\{x : \Omega(x,u) \leqq \alpha\}$, $\alpha > 0$. Here

$$\{x : \Omega(x,u) \leqq \alpha\} = \left\{x : \Omega\left(x, \frac{u}{\alpha}\right) \leqq 1\right\} = \mathbb{L}\left(\frac{u}{\alpha}\right) \text{ (Closed)}$$

since $u \to \mathbb{L}(u)$ is the inverse correspondence of $x \to \mathbb{P}(x)$. Thus $\Omega(x,u)$ is lower semi-continuous in x.

By entirely analogous definition and argument it is seen that the distance functional is also lower semi-continuous in u for $x \in (L_\infty)^n_+$.

4.2 The Input Distance Functional

For $u \in (L_\infty)^m_+$ and $x \in (L_\infty)^n_+$, the input distance functional $\underline{\bar{\Psi}} : (L_\infty)^m_+ \times (L_\infty)^n_+ \to R_+$ is defined (Shephard, 1970:a) by

$$\text{(4.2-1)} \qquad \underline{\bar{\Psi}}(u,x) = [\text{Min } \{\lambda \in R_+ : (\lambda x) \in \mathbb{L}(u)\}]^{-1} .$$

This functional provides a means of defining the correspondence $u \to \mathbb{L}(u)$. When $u = 0$, $\mathbb{L}(0) = (L_\infty)^n_+$ and $\underline{\bar{\Psi}}(0,x) = +\infty$ for all $x \in (L_\infty)^n_+$. If $u \geq 0$ and $\mathbb{L}(u)$ is empty, $\text{Min } \{\lambda \in R_+ : (\lambda x) \in \emptyset\} = +\infty$ and $\underline{\bar{\Psi}}(u,x) = 0$ for all $x \in (L_\infty)^n_+$. When $u \geq 0$ and $\mathbb{L}(u) \neq \emptyset$, there are two cases to consider depending upon whether the intersection $\{\lambda x : \lambda \in [0,+\infty)\} \cap \mathbb{L}(u)$ is empty or not. If $x = 0$ the intersection is empty and $\underline{\bar{\Psi}}(u,0) = 0$. If $x \neq 0$ and the intersection is empty, likewise $\underline{\bar{\Psi}}(u,x) = 0$. However, if $x \in (L_\infty)^n_+$, $x \neq 0$ and the intersection is not empty, $0 < \underline{\bar{\Psi}}(u,x) < +\infty$, and, if $x \in \mathbb{L}(u)$, $\underline{\bar{\Psi}}(u,x) \geqq 1$.

By the cases outlined it is clearly that if $\bar{\underline{\Psi}}(u,x) \geqq 1$, $x \varepsilon \mathbb{L}(u)$. Hence the following proposition holds:

<u>Proposition (4.2-1)</u>: (Shephard, 1970:a)

$x \varepsilon \mathbb{L}(u)$ if and only if $\bar{\underline{\Psi}}(u,x) \geqq 1$, i.e., $\mathbb{L}(u) = \left\{x \varepsilon (L_\infty)^n_+ : \bar{\underline{\Psi}}(u,x) \geqq 1\right\}$, $u \varepsilon (L_\infty)^m_+$.

The isoquant of $\mathbb{L}(u)$ is defined by

$$(4.2\text{-}2) \quad \mathrm{ISOQ}\,\mathbb{L}(u) := \begin{cases} \{x \varepsilon \mathbb{L}(u) : (\lambda x) \notin \mathbb{L}(u) \ , \ \lambda \varepsilon [0,1)\} \ , u \neq 0 \ , \ \mathbb{L}(u) \neq \emptyset \\ \{0\} \ , \ u = 0 \end{cases}$$

and, with this definition, the following proposition holds:

<u>Proposition (4.2-2)</u>: (Shephard, 1970:a)

$x \varepsilon \mathrm{ISOQ}\ \mathbb{L}(u)$ if and only if $\bar{\underline{\Psi}}(u,x) = 1$.

The definition of isoquant given by expression (4.2-1) is a dynamic extension of the familiar notion of a steady state (static) isoquant defining the factor substitutions possible for attaining a given output vector u . Here, the substitutions defined by the functional equation

$$(4.2\text{-}3) \qquad \bar{\underline{\Psi}}(u,x) = 1$$

are both factor substitutions between and time substitutions within and between the time histories of the output rates of the factors. The alternatives so defined are considerably more complicated than those in the steady state (static) case. In the dynamic context the input isoquant is composed of those vectors of input rate histories yielding u which

cannot be homogeneously scaled downward for u, preserving the time distributions of input rates.

By arguments which exactly parallel those given in the case of the output distance functional, it is easy to see that the distance functional $\underline{\bar{\Psi}}(u,x)$ has the properties stated in the following proposition:

Proposition (4.2-3): (Shephard, 1970:a)

The distance functional $\underline{\bar{\Psi}}(u,x)$ satisfies:

$\underline{\bar{\Psi}}.1$ $\underline{\bar{\Psi}}(u,x) = 0$ for $\mathbb{L}(u) = \emptyset$ and $x \in (L_\infty)^n_+$, or $\mathbb{L}(u) \neq \emptyset$ with $\{\lambda x : \lambda \in [0,+\infty)\} \cap \mathbb{L}(u) = \emptyset$, and $\underline{\bar{\Psi}}(0,x) = +\infty$ for $x \in (L_\infty)^n_+$.

$\underline{\bar{\Psi}}.2$ $\underline{\bar{\Psi}}(u,x) \geqq 0$ and $\underline{\bar{\Psi}}(u,x) > 0$ for $\mathbb{L}(u) \neq \emptyset$ with $\{\lambda x : \lambda \in [0,+\infty)\} \cap \mathbb{L}(u) \neq \emptyset$.

$\underline{\bar{\Psi}}.3$ $\underline{\bar{\Psi}}(u,\lambda x) = \lambda\underline{\bar{\Psi}}(u,x)$, $\lambda \in (0,+\infty)$, $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$.

$\underline{\bar{\Psi}}.4$ $\underline{\bar{\Psi}}(\theta u,x) \leqq \underline{\bar{\Psi}}(u,x)$ for $\theta \geqq 1$, $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$.

$\underline{\bar{\Psi}}.5$ $\underline{\bar{\Psi}}(u,x)$ is upper semi-continuous in $x \in (L_\infty)^n_+$ and in $u \in (L_\infty)^m_+$.

Here the upper semi-continuity of $\underline{\bar{\Psi}}(u,x)$ in $x \in (L_\infty)^n_+$, say, is equivalent to

$$\left\{x \in (L_\infty)^n_+ : \underline{\bar{\Psi}}(u,x) \geqq \alpha\right\}$$

being closed for all real numbers α. By using the stronger properties $\mathbb{L}$.6S or $\mathbb{L}$.6SS property $\underline{\bar{\Psi}}$.4 may be strengthened to:

$\underline{\bar{\Psi}}.4S$ $\underline{\bar{\Psi}}(\theta_1 u_1, \ldots, \theta_m u_m, x) \leqq \underline{\bar{\Psi}}(u,x)$ for $\theta_i \in [1,+\infty)$,

$i \in \{1,2, \ldots, m\}$,

$\underline{\bar{\Psi}}.4SS$ $\underline{\bar{\Psi}}(u',x) \leqq \underline{\bar{\Psi}}(u,x)$ for $u' \leqq u$.

4.3 The Joint Production Functional

A Joint Production Functional (J.P.F.) is defined by:

Definition (4.3-1): (Shephard, 1970:a)

A functional $\mathbb{E} : (L_\infty)^n_+ \times (L_\infty)^m_+ \to R_+$ with the properties:

(a) For $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, ISOQ $\mathbb{L}(u) = \left\{x \in (L_\infty)^n_+ : \mathbb{E}(x,u) = 0\right\}$.

(b) For $x \geq 0$, $\mathbb{P}(x) \neq \{0\}$, ISOQ $\mathbb{P}(x) = \left\{u \in (L_\infty)^m_+ : \mathbb{E}(x,u) = 0\right\}$

is a J.P.F.

Proposition (4.3-1): (Bol and Moeschlin, 1975)

A J.P.F. $\mathbb{E}(x,u)$ exists if and only if for all $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, $\mathbb{P}(x) \neq \{0\}$, $x \in$ ISOQ $\mathbb{L}(u) \Longleftrightarrow u \in$ ISOQ $\mathbb{P}(x)$.

Suppose $\mathbb{E}(x,u)$ is a J.P.F. and that $\mathbb{E}(x,u) = 0$. Clearly $x \in$ ISOQ $\mathbb{L}(u) \Longleftrightarrow u \in$ ISOQ $\mathbb{P}(x)$. Conversely suppose $x \in$ ISOQ $\mathbb{L}(u) \Longleftrightarrow u \in$ ISOQ $\mathbb{P}(x)$ and define

$$M := \{(x,u) : x \geq 0 , u \geq 0 , \mathbb{P}(x) \neq \{0\} , u \in \text{ISOQ } \mathbb{P}(x)\} =$$
$$\{(x,u) : x \geq 0 , u \geq 0 , \mathbb{L}(u) \neq \emptyset , x \in \text{ISOQ } \mathbb{L}(u)\} .$$

Let

$$(4.3\text{-}1) \qquad \mathbb{E}(x,u) := \begin{cases} 0 & \text{for } (x,u) \in M \\ 0 & \text{for } u = 0 \text{ and } \mathbb{P}(x) = \{0\} \\ 1 & \text{otherwise.} \end{cases}$$

The functional $\mathbb{E}(x,u)$ given by (4.3-1) is a J.P.F.

The construction of a J.P.F. can be made in terms of the distance functionals $\Omega(x,u)$, $\bar{\underline{\Psi}}(u,x)$. The following proposition verifies this fact.

Proposition (4.3-2): (Shephard, 1970:a)

If a J.P.F. exists it may be expressed by $\mathbb{E}(x,u) = A(\bar{\underline{\Psi}}(u,x) - \Omega(x,u))$ where $A(\cdot)$ is any monotone transformation with $A(0) = 0$.

Suppose that a J.P.F. exists. Then $x \in \text{ISOQ } \mathbb{L}(u) \Longleftrightarrow u \in \text{ISOQ } \mathbb{P}(x)$. Consider

$$\bar{\mathbb{E}}(x,u) = (\bar{\underline{\Psi}}(u,x) - \Omega(x,u)) .$$

If $x \in \text{ISOQ } \mathbb{L}(u)$, $\bar{\underline{\Psi}}(u,x) = 1 = \Omega(x,u)$ and $\bar{\mathbb{E}}(x,u) = 0$. Suppose $\bar{\mathbb{E}}(x,u) = 0$ and $\bar{\underline{\Psi}}(u,x) \neq 1$. Then either $\bar{\underline{\Psi}}(u,x) < 1$ or $\bar{\underline{\Psi}}(u,x) > 1$. In the first case $x \notin \mathbb{L}(u)$ implying $u \notin \mathbb{P}(x)$ which implies $\Omega(x,u) > 1$, a contradiction. Similarly $\bar{\underline{\Psi}}(u,x) > 1$ leads to a contradiction. Thus $\bar{\mathbb{E}}(x,u)$ is a J.P.F., and any monotone transformation of $\bar{\mathbb{E}}(x,u)$ is a J.P.F.

Sub-Proposition (4.3-2):

The J.P.F. is not unique.

A necessary and sufficient condition for the existence of a J.P.F. is given by:

Proposition (4.3-3): (Al-Ayat and Färe, 1977)

For $x \geq 0$, $u \geq 0$, $\mathbb{P}(x) \neq \{0\}$ and $\mathbb{L}(u) \neq \emptyset$, a necessary and sufficient condition for the existence of a J.P.F. is

$$(4.3\text{-}2) \quad \text{ISOQ } \mathbb{P}(x) \cap \text{ISOQ } \mathbb{P}(\lambda x) = \text{ISOQ } \mathbb{L}(u) \cap \text{ISOQ } \mathbb{L}(\theta u) = \emptyset$$

for all positive scalars λ , $\theta \neq 1$.

In order to show (4.3-2) necessary, assume that a J.P.F. exists and suppose for $\lambda \neq 1$ that $u \in \text{ISOQ } \mathbb{P}(x) \cap \text{ISOQ } \mathbb{P}(\lambda x)$. Then by Proposition (4.3-1), $x \in \text{ISOQ } \mathbb{L}(u)$ and $(\lambda x) \in \text{ISOQ } \mathbb{L}(u)$, $\lambda \neq 1$, a contradiction. Similarly one may dispose of the case where $x \in \text{ISOQ } \mathbb{L}(u) \cap \text{ISOQ } \mathbb{L}(\theta u)$, $\theta \neq 1$.

For the sufficiency assume that relation (4.3-2) holds. Also suppose that a J.P.F. does not exist. Then for some $x \in (L_\infty)^n_+$, $u \in (L_\infty)^n_+$, $x \geq 0$, $u \geq 0$, $\mathbb{P}(x) \neq \{0\}$ and $\mathbb{L}(u) \neq \emptyset$, $u \in \text{ISOQ } \mathbb{P}(x) \not\Rightarrow x \in \text{ISOQ } \mathbb{L}(u)$. Then, since $u \in \text{ISOQ } \mathbb{P}(x) \Rightarrow x \in \mathbb{L}(u)$ there exists a positive scalar $\lambda < 1$ such that $(\lambda x) \in \text{ISOQ } \mathbb{L}(u)$, implying $u \in \mathbb{P}(\lambda x) \subset \mathbb{P}(x)$ for $\lambda < 1$. By (4.3-2) it follows that $u \notin \text{ISOQ } \mathbb{P}(x)$, a contradiction. A similar argument shows that $\text{ISOQ } \mathbb{L}(u) \cap \text{ISOQ } \mathbb{L}(\theta u) = \emptyset$ for $\theta \neq 1$ is likewise sufficient for the existence of a J.P.F.

When the condition (4.3-2) is taken with properties Ω.4 and $\underline{\bar{\Psi}}$.5 it follows that:

Sub-Proposition (4.3-3):

A J.P.F. exists for $x \geq 0$, $u \geq 0$, $\mathbb{P}(x) \neq \{0\}$, $\mathbb{L}(u) \neq \emptyset$, if and only if the two distance functionals $\Omega(x,u)$ and $\underline{\bar{\Psi}}(u,x)$ are strictly decreasing for scaling x and u respectively.

Further regularity concerning the behavior of the correspondence $x \to \mathbb{P}(x)$ and $u \to \mathbb{L}(u)$ is implied if a J.P.F. exists.

Definition (4.3-2):

The correspondence $x \to \mathbb{P}(x)$ is continuous along rays if and only if

$$\mathbb{P}(\lambda^{o}x) = \text{CLOSURE} \bigcup_{0 \leqq \lambda < \lambda^{o}} \mathbb{P}(\lambda x) \;, \; x \in (L_\infty)^n_+ \;.$$

Definition (4.3-3):

The correspondence $u \to \mathbb{L}(u)$ is continuous along rays if and only if

$$\mathbb{L}(\theta^{o}u) = \text{CLOSURE} \bigcup_{\theta > \theta^{o}} \mathbb{L}(\theta u) \;, \; u \in (L_\infty)^m_+ \;, \; \mathbb{L}(u) \neq \emptyset \;.$$

Proposition (4.3-4): (Bol and Moeschlin, 1975)

If there exists a J.P.F., the correspondences $x \to \mathbb{P}(x)$ and $u \to \mathbb{L}(u)$ are continuous along rays.

Suppose that $x \to \mathbb{P}(x)$ is discontinuous at $x^{o} \in (L_\infty)^n_+$, $x^{o} \geq 0$, $\mathbb{P}(x^{o}) \neq \{0\}$. Then $\mathbb{P}(x^{o}) \neq \text{CLOSURE} \bigcup_{0 \leqq \lambda < \lambda^{o}} \mathbb{P}(\lambda x) := \bar{\mathbb{P}}(x^{o})$, with $\bar{\mathbb{P}}(x^{o}) \subset \mathbb{P}(x^{o})$. Then there exists a vector $u^{o} \in \text{ISOQ } \mathbb{P}(x^{o})$ with $u^{o} \notin \bar{\mathbb{P}}(x^{o})$. Since $\bar{\mathbb{P}}(x^{o})$ is closed, there exists a neighborhood N of u^{o} such that $N \cap \bar{\mathbb{P}}(x^{o}) = \emptyset$. Hence there is a positive scalar $\lambda < 1$ such that $(\lambda u^{o}) \notin \bar{\mathbb{P}}(x^{o})$, since $\bar{\mathbb{P}}(x^{o}) \subset \mathbb{P}(x^{o})$ it follows that $(\lambda u^{o}) \notin \mathbb{P}(x^{o})$ for $0 < \lambda < 1$, implying $x^{o} \notin \mathbb{L}(\lambda u^{o}) \supset \mathbb{L}(u^{o})$ and $x^{o} \notin \mathbb{L}(u^{o})$, whence $x^{o} \notin \text{ISOQ } \mathbb{L}(u^{o})$. Then by Proposition (4.3-1) it follows that a J.P.F. cannot exist. Similarly, if $u \to \mathbb{L}(u)$ is discontinuous at u^{o}, it follows that a J.P.F. cannot exist.

Sub-Proposition (4.3-4): (Shephard, 1970:a)

If there exists a J.P.F., the distance functionals $\Omega(x,u)$ and $\bar{\Psi}(u,x)$ are continuous along rays with respect to x and u respectively.

As an example of the use of Proposition (4.3-2) to construct a J.P.F., consider the example of a dynamic neoclassical production function (2.3-20)

$$u_\nu := \begin{cases} \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}} & \text{for } x \in X,\ \nu = 1, \ldots, N+1 \\ 0 \text{ for } x \notin X . \end{cases}$$

This expression is homogeneous of degree $+1$ and therefore strictly increasing and continuous along rays, so that a J.P.F. exists (see Proposition (4.3-3)). With this dynamic production function, consider

$$\mathbb{P}(x) := \left\{ u : u_\nu \leqq \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}},\ \nu = 1, 2, \ldots, (N+1) \right\} .$$

$$\mathbb{L}(u) := \left\{ x \in X : \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}} \geqq u_\nu ,\ \nu = 1, 2, \ldots, (N+1) \right\} .$$

The distance functionals for these two correspondences are calculated by:

$$\Omega(x,u) = \left[\operatorname*{Min}_{\nu} \left(\operatorname{Max} \left\{ \theta_\nu : \theta_\nu u_\nu \leqq \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}} \right\} \right) \right]^{-1}$$

$$= \left[\operatorname*{Min}_{\nu} \theta^{*}_{\nu} \right]^{-1}$$

where

$$\theta_\nu^* = \frac{\left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}}}{u_\nu}, \quad \nu = 1,2, \ldots, (N+1) ,$$

$$\underline{\bar{\Psi}}(u,x) = \left[\operatorname*{Max}_{\nu}\left(\operatorname{Min}\left\{\lambda_\nu : \lambda_\nu \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}} \geqq u\right\}\right)\right]^{-1}$$

$$= \left[\operatorname*{Max}_{\nu} \lambda_\nu^*\right]^{-1}$$

where

$$\lambda_\nu^* = \frac{u_\nu}{\left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho}\right]^{-\frac{1}{\rho}}}, \quad \nu = 1,2, \ldots, (N+1) .$$

A J.P.F. is then expressed by

$$\mathbb{E}(x,u) = \left[\left(\operatorname*{Max}_{\nu} \lambda_\nu^*\right)^{-1} - \left(\operatorname*{Min}_{\nu} \theta_\nu^*\right)^{-1}\right] .$$

Some appreciation of the possibilities for ISOQ $\mathbb{P}(x)$ can be obtained by noting that, if ν^* denotes an integer for which θ_ν^* attains the maximal value, any function of the form

$$u_\nu^* = \left[\sum_{j=1}^{n} \beta_{j\nu}^*(X_{j\nu}^*)^{-\rho}\right]^{-\frac{1}{\rho}}$$

$$u_\nu \leqq \left[\sum_{j=1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho} \right]^{-\frac{1}{\rho}}, \quad \nu \in \{1,2, \ldots, (N+1)\} \sim \{\nu^*\}$$

belongs to ISOQ $\mathbb{P}(x)$. Thus one may see the complexity of the output set $\mathbb{P}(x)$.

CHAPTER 5

SPECIAL STRUCTURES

Certain special structures are useful in the development of a general theory, particularly those related to scaling of production. Special forms for the dynamic production correspondence of this type will be discussed in this chapter. The important class of production correspondences known as Activity Analysis Models will be discussed at length in later chapters to develop applications of the dynamic theory and related computational methods.

5.1 Globally Homothetic Correspondences

The notion of a homothetic production correspondence (function) introduced by (Shephard, 1953) and extended in (Shephard, 1970:a) has found wide application in production theory. In the dynamic context this property is expressed by:

Definition (5.1-1):

A dynamic production correspondence $\mathbb{P} : x \in (L_\infty)^n_+ \to \mathbb{P}(x) \in 2^{(L_\infty)^m_+}$ is Globally Homothetic if and only if $\mathbb{P}(x) = F(\mathbb{H}(x)) \cdot \mathbb{P}_{ff}(1)$.
In this definition, certain prescriptions are required for the function $F : v \in R_+ \to F(v) \in R_+$, the fixed set $\mathbb{P}_{ff}(1) \subset (L_\infty)^m_+$ and the functional $\mathbb{H} : x \in (L_\infty)^n_+ \to \mathbb{H}(x) \in R_+$, in order that the correspondence $x \to \mathbb{P}(x)$ so structured satisfy the axioms for a production correspondence.
As stated below, these prescriptions are taken so that the weak forms of the axioms are fulfilled:

(a) $F(\cdot)$ is nonnegative, nondecreasing, real valued function which is upper semi-continuous with $F(0) = 0$ and $F(v) \to +\infty$ as $v \to +\infty$.

(b) $\mathbb{P}_{\mathbb{ff}}(1)$ is a closed and bounded fixed subset of vectors of $(L_\infty)_+^m$ satisfying $\mathbb{P}$.T.1 and $\mathbb{P}$.T.2, with $0 \in \mathbb{P}_{\mathbb{ff}}(1)$ and $(\theta u) \in \mathbb{P}_{\mathbb{ff}}(1)$ for $\theta \in [0,1]$ when $u \in \mathbb{P}_{\mathbb{ff}}(1)$.

(c) $\mathbb{H}(x)$ satisfies:

$\mathbb{H}$.1 $\mathbb{H}(0) = 0$, $\mathbb{H}(x) > 0$ for some $x \in (L_\infty)_+^n$.

$\mathbb{H}$.2 $\mathbb{H}(x) < +\infty$ for $||x|| < +\infty$.

$\mathbb{H}$.3 $\mathbb{H}(\lambda x) \geqq \mathbb{H}(x)$ for $\lambda \in [1,+\infty)$.

$\mathbb{H}$.4 If $\mathbb{H}(x) > 0$, $\mathbb{H}(\lambda x) \to +\infty$ as $\lambda \to +\infty$.

$\mathbb{H}$.5 $\mathbb{H}(x)$ is upper semi-continuous on $(L_\infty)_+^n$.

For stronger axioms one may take $\mathbb{P}_{\mathbb{ff}}(1)$ totally bounded with

$(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \in \mathbb{P}_{\mathbb{ff}}(1)$ for $\theta_i \in [0,1]$, $i \in \{1,2, \ldots, m\}$

when $u \in \mathbb{P}_{\mathbb{ff}}(1)$ or

$$u' \in \mathbb{P}_{\mathbb{ff}}(1) \text{ for } 0 \leqq u' \leqq u \text{ when } u \in \mathbb{P}_{\mathbb{ff}}(1) ,$$

and replace $\mathbb{H}$.3 by

$$\mathbb{H}(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) \geqq \mathbb{H}(x) \text{ for } \lambda_i \in [1,+\infty) , i \in \{1,2, \ldots, n\}$$

or

$$\mathbb{H}(x') \geqq \mathbb{H}(x) \text{ for } x' \geqq x .$$

In a similar way, a homothetic dynamic input correspondence is defined by:

<u>Definition (5.1-2):</u>

A dynamic production correspondence $\mathbb{L} : u \in (L_\infty)_+^m \to \mathbb{L}(u) \in 2^{(L_\infty)_+^n}$ is Globally Homothetic if and only if $\mathbb{L}(u) = G(\mathbb{J}(u)) \cdot \mathbb{L}_{\phi}(1)$ for $u \geq 0$ with:

(a) $G : v \in R_+ \to G(v) \in R_+$ is nonnegative, nondecreasing, lower semi-continuous, with $G(v) \to +\infty$ as $v \to +\infty$.

(b) $\mathbb{L}_{\phi}(1)$ is a closed, fixed subset of vectors of $(L_\infty)_+^n$ satisfying $\mathbb{L}$.T.1 and $\mathbb{L}$.T.2, with $(\lambda x) \in \mathbb{L}_{\phi}(1)$ for $\lambda \in [1,+\infty)$ when $x \in \mathbb{L}_{\phi}(1)$, and $\{x \in \mathbb{L}_{\phi}(1) : y \notin \mathbb{L}_{\phi}(1)$ for $y \leq x\}$ is bounded.

(c) $\mathbb{J} : u \in \left((L_\infty)_+^m \sim 0\right) \to \mathbb{J}(u) \in R_+$ is a functional satisfying:

$\mathbb{J}$.1 $\mathbb{J}(u) > 0$ for some $u \in \left((L_\infty)_+^m \sim 0\right)$.

$\mathbb{J}$.2 $\mathbb{J}(u) < +\infty$ for $||u|| < +\infty$ and $\mathbb{L}(u) \neq \emptyset$, and $\mathbb{J}(u) = +\infty$ for $\mathbb{L}(u) = \emptyset$.

$\mathbb{J}$.3 $\mathbb{J}(\theta u) \geqq \mathbb{J}(u)$ for $\theta \in [1,+\infty)$.

$\mathbb{J}$.4 If $\mathbb{J}(u) > 0$, $\mathbb{J}(\theta u) \to +\infty$ as $\theta \to +\infty$.

$\mathbb{J}$.5 $\mathbb{J}(u)$ is lower semi-continuous on $\left((L_\infty)_+^m \sim 0\right)$.

To conform with stronger axioms for the correspondence $u \to \mathbb{L}(u)$, one may take

$$\{x \in \mathbb{L}_{\phi}(1) : y \notin \mathbb{L}_{\phi}(1) \text{ for } y \leq x\} \text{ totally bounded,}$$

$$(\lambda x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) \in \mathbb{L}_{\phi}(1) \text{ for } \lambda_i \in [1,+\infty) \text{ , } i \in \{1,2, \ldots, n\}$$

when $x \in \mathbb{L}_{\phi}(1)$ or

$$x' \in \mathbb{L}_{\phi}(1) \text{ when } x' \geqq x \text{ and } x \in \mathbb{L}_{\phi}(1) ,$$

and replace $\mathbb{J}.3$ by

$$\mathbb{J}(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \geqq \mathbb{J}(u) \text{ for } \theta_i \in [1,+\infty) , i \in \{1,2, \ldots, m\}$$

or

$$\mathbb{J}(u') \geqq \mathbb{J}(u) \text{ for } u' \geqq u .$$

Although these special structures are defined for inversely related production correspondences, neither implies the other. A special functional relationship defines the correspondences in each case, as expressed by the distance functionals. (See Chapter 4.)

The distance functional for $\mathbb{P}_{\mathbb{F}}(1)$ is expressed by

$$(5.1\text{-}1) \quad \mathbb{F}(u) := [\text{Max } \{\theta : (\theta u) \in \mathbb{P}_{\mathbb{F}}(1) , \theta \in [0,+\infty)\}]^{-1}$$

for $\Omega(x,u)$, since $x \in (L_\infty)^n_+$ is omitted because $\mathbb{P}_{\mathbb{F}}(1)$ is a fixed set. Similarly the distance functional for $\mathbb{L}_{\phi}(1)$ is expressed by

$$(5.1\text{-}2) \quad \phi(x) := [\text{Min } \{\lambda : (\lambda x) \in \mathbb{L}_{\phi}(1) , \lambda \in [0,+\infty)\}]^{-1} .$$

Note that $\mathbb{F}(u)$ and $\phi(x)$ are homogeneous of degree $+1$. Then

$$\mathbb{P}(x) = F(\mathbb{H}(x))\mathbb{P}_{\mathbb{f}}(1)$$

$$(5.1\text{-}3) \qquad = F(\mathbb{H}(x)) \cdot \left\{u \in (L_\infty)^m_+ : \mathbb{f}(u) \leqq 1\right\}$$

$$= \left\{u \in (L_\infty)^m_+ : \mathbb{f}(u) \leqq F(\mathbb{H}(x))\right\}$$

and similarly

$$(5.1\text{-}4) \qquad \mathbb{L}(u) = \left\{x \in (L_\infty)^n_+ : \phi(x) \geqq G(\mathbb{J}(u))\right\} .$$

Clearly, it follows then that

$$(5.1\text{-}5) \qquad \text{ISOQ } \mathbb{P}(x) = \left\{u \in (L_\infty)^m_+ : \mathbb{f}(u) = F(\mathbb{H}(x))\right\} .$$

$$(5.1\text{-}6) \qquad \text{ISOQ } \mathbb{L}(u) = \left\{x \in (L_\infty)^n_+ : \phi(x) = G(\mathbb{J}(u))\right\} .$$

Suppose, as a further specialization, that the functional $\mathbb{H}(x)$ of Definition (5.1-1) is homogeneous of degree $+1$, say $\mathbb{H}(x) = \phi(x)$, with $\phi(\lambda x) = \lambda\phi(x)$. Then the global homotheticity of $x \to \mathbb{P}(x)$ implies

$$\mathbb{L}(u) = \left\{x \in (L_\infty)^n_+ : u \in \mathbb{P}(x)\right\}$$

$$= \left\{x \in (L_\infty)^n_+ : \mathbb{f}(u) \leqq F(\phi(x))\right\}$$

$$= \left\{x \in (L_\infty)^n_+ : \phi(x) \geqq F^{-1}(\mathbb{f}(u))\right\}$$

where

$$F^{-1}(w) := \text{Min}\ \{v \in R_+ : F(v) \geqq w\}\ ,\ w \in R_+ .$$

Then

$$(5.1\text{-}7) \qquad \mathbb{L}(u) = F^{-1}(\mathbb{F}(u))\left\{x \in (L_\infty)^n_+ : \Phi(x) \geqq 1\right\} .$$

Further, $\left\{x \in (L_\infty)^n_+ : \Phi(x) \geqq 1\right\}$ is a fixed subset of $(L_\infty)^n_+$ with distance functional $\Phi(x)$, and

$$\mathbb{L}(u) = F^{-1}(\mathbb{F}(u)) \cdot \mathbb{L}_\Phi(1) .$$

Thus homogeneity of $\mathbb{H}(x)$ for the global homotheticity of $x \to \mathbb{P}(x)$ implies that $u \to \mathbb{L}(u)$ is globally homothetic. Conversely, if the functional $\mathbb{J}(u)$ of Definition (5.1-2) is homogeneous of degree + 1 for $u \to \mathbb{L}(u)$ to be globally homothetic, the inverse correspondence $x \to \mathbb{P}(x)$ is globally homothetic.

<u>Proposition (5.1-1)</u>: (Shephard, 1970:a)

If $x \to \mathbb{P}(x)$ $(u \to \mathbb{L}(u))$ is globally homothetic with $\mathbb{H}(\lambda x) = \mathbb{H}(x)$, $\lambda > 0$, $(\mathbb{J}(\theta u)) = \theta \mathbb{J}(u)$, $\theta > 0)$ the inverse correspondence $u \to \mathbb{L}(u)$ $(x \to \mathbb{P}(x))$ is globally homothetic.

A special and interesting case of global homotheticity occurs when

$$(5.1\text{-}8) \qquad \begin{aligned} \mathbb{P}(x) &: = F(\Phi(x)) \cdot \mathbb{P}_{\mathbb{F}}(1) \\ \mathbb{L}(u) &: = F^{-1}(\mathbb{F}(u)) \cdot \mathbb{L}_\Phi(1) \end{aligned}$$

where $\Phi(x)$ and $\mathbb{F}(u)$ are distance functionals for $\mathbb{L}_\Phi(1)$ and $\mathbb{P}_{\mathbb{F}}(1)$ respectively, i.e., the two correspondences are Inversely Homothetic. Then it is clear from the foregoing (see 5.1.3 and 5.1.4) that

$$\mathbb{E}(x,u) := [\mathbb{f}(u) - F(\phi(x))] \tag{5.1-9}$$

is a J.P.F., when $F(\cdot)$ is strictly increasing.

As an example of inversely related homothetic production correspondences, consider the distance functionals

$$\bar{\Psi}(1,x) := \begin{cases} \phi(x) = \left[\underset{\nu}{\mathrm{Max}} \left(\dfrac{1}{\left[\sum_{j=1}^{n} \beta_{j\nu} (X_{j\nu})^{-\rho} \right]^{-\frac{1}{\rho}}} \right) \right]^{-1}, & x \in X \\ +\infty, & x \notin X \end{cases} \tag{5.1-10}$$

$$\Omega(1,u) := \mathbb{f}(u) = \left[\underset{\nu}{\mathrm{Min}} \left(\dfrac{\left[\sum_{j=1}^{n} \beta_{j\nu} (X_{j\nu}(1))^{-\rho} \right]^{-\frac{1}{\rho}}}{u_\nu} \right) \right]^{-1}, \tag{5.1-11}$$

where $X_{j\nu}(1)$ is as defined by (2.3-16) with $x_j(\tau) = 1$ for $j \in \{1,2, ..., n\}$ and all values of $\tau \geqq 1$, and u is taken constantly equal to 1 in the definition of $\bar{\Psi}(1,x)$. Then 5.1-8 defines inversely related homothetic correspondences in these terms when

$$\mathbb{P}_{\mathbb{f}}(1) := \left\{ u \in (L_\infty)^m_+ : \mathbb{f}(u) \leqq 1 \right\}$$

$$\mathbb{L}_{\phi}(1) := \left\{ x \in (L_\infty)^n_+ : \phi(x) \geqq 1 \right\}.$$

The J.P.F. for the special structure takes the form (5.1-9) with $\phi(x)$ and $\mathbb{f}(u)$ given by (5.1-10), (5.1-11) respectively.

5.2 Semi-Homogeneous Correspondences

Another structure of some interest for scaling of production is one where for given input vector $x \in (L_\infty)^n_+$ a scaling of x results in homogeneous scaling of the output set $\mathbb{P}(x)$, but the degree of homogeneity depends upon the mix of the vector x. An output correspondence of this type is defined by:

Definition (5.2-1): (Shephard, 1974:b)

The dynamic production correspondence $\mathbb{L} : u \in (L_\infty)^m_+ \to \mathbb{L}(u) \in 2^{(L_\infty)^n_+}$ is Semi-Homogeneous if and only if $\mathbb{L}(\theta u) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \mathbb{L}(u)$, $\theta \in (0,+\infty)$, $\mathbb{L}(u) \neq \emptyset$, where $\mathbb{B} : u \in (L_\infty)^m_+ \to \mathbb{B}\left(\frac{u}{||u||}\right) \in R_+$ is a positive, bounded functional for $u \in \left\{u \in (L_\infty)^m_+ : ||u|| = 1\right\}$, constant along rays in $(L_\infty)^m_+$.

Clearly the semi-homogeneous dynamic correspondence $u \to \mathbb{L}(u)$ satisfies the weak axiom $\mathbb{L}.3$. Further, the correspondence may be taken to satisfy any of the other axioms for $\mathbb{L}(u)$ (see Section 2.1.1). However, if certain strong axioms are used, the property of semi-homogeneity degenerates into simple homogeneity. In the development of this degeneration, the following lemma will be used:

Lemma (5.2-1): (Shephard, 1974:b)

If the dynamic production correspondence $u \to \mathbb{L}(u)$ is semi-homogeneous, $x \in \text{ISOQ}\ \mathbb{L}(u) \iff u \in \text{ISOQ}\ \mathbb{P}(x)$.

Verification of this lemma is simple. If $x \in \text{ISOQ}\ \mathbb{L}(u)$ and $\mathbb{L}(\theta u) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u)$, then $\lambda x \notin \mathbb{L}(u)$ for $\lambda \in (0,1)$ and $x \notin \frac{1}{\lambda}\mathbb{L}(u) = \mathbb{L}\left(\lambda^{-\left(\mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)} \cdot u\right)$, implying for $\sigma := \lambda^{-\left(\mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)}$

that $\sigma u \notin \mathbb{P}(x)$ for $\sigma \in (1,+\infty)$, and $u \in \text{ISOQ } \mathbb{P}(x)$. By reverse construction $u \in \text{ISOQ } \mathbb{P}(x) \Rightarrow x \in \text{ISOQ } \mathbb{L}(u)$.

To investigate the consequences of free disposability of outputs, i.e., $\mathbb{P}$.6SS, for a semi-homogeneous dynamic correspondence $u \to \mathbb{L}(u)$, consider first $u \in \left\{u \in (L_\infty)^m_+ : u >>> 0\right\} : = (L_\infty)^m_{++}$. Axiom $\mathbb{P}$.4.1 does not guarantee the existence of $u \in (L_\infty)^m_{++}$ such that $\mathbb{L}(u) \neq \emptyset$. However, if $\mathbb{L}(u) \neq \emptyset$ for $u \in (L_\infty)^m_{++}$, it is implied by $\mathbb{P}$.6SS and $\mathbb{P}$.4.2 that $\mathbb{L}(u) \neq \emptyset$ for all $u \in (L_\infty)^m_+$. In the present context it is supposed for $u \in (L_\infty)^m_{++}$ that $\mathbb{L}(u) \neq \emptyset$. Then by axiom $\mathbb{P}$.6SS there exists a vector $v \in (L_\infty)^m_{++}$ such that $v \neq \theta u$ for $\theta \in (0,+\infty)$, $v <<< u$, and $v \in \mathbb{P}(\bar{x})$ where $\bar{x} \in \mathbb{L}(u) \Longleftrightarrow u \in \mathbb{P}(\bar{x})$. Consider the scaled vectors

$$\bar{u} : \bar{\theta} u \in \text{ISOQ } \mathbb{P}(\bar{x}) \text{ , } \bar{v} : = \bar{\sigma} v \in \text{ISOQ } \mathbb{P}(\bar{x}) \text{ .}$$

By Lemma (5.2-1), $\bar{x} \in \text{ISOQ } \mathbb{L}(\bar{u})$, $\bar{x} \in \text{ISOQ } \mathbb{L}(\bar{v})$, and for $\lambda \in (0,+\infty)$

$$w(\lambda) : = \theta_\lambda \bar{u} \in \text{ISOQ } \mathbb{P}(\lambda\bar{x}) \text{ , } z(\lambda) : = \sigma_\lambda \bar{v} \in \text{ISOQ } \mathbb{P}(\bar{\lambda} x)$$

where

$$\theta_\lambda : = \lambda^{\mathbb{B}\left(\frac{u}{||u||}\right)^{-1}} \text{ , } \sigma_\lambda : = \lambda^{\mathbb{B}\left(\frac{v}{||v||}\right)^{-1}} \text{ .}$$

Suppose $\mathbb{B}\left(\frac{u}{||u||}\right) > \mathbb{B}\left(\frac{v}{||v||}\right)$. Let α denote the component identifying index of $\bar{u}$, $\bar{v}$ such that .

$$\underline{S}_\alpha : = \frac{||\bar{v}_\alpha||}{||\bar{u}_\alpha||} = \underset{i}{\text{Min}} \left\{ \frac{||\bar{v}_i||}{||\bar{u}_i||} : i \in \{1,2, \ldots, m\} \right\} \text{ .}$$

Now, for $\lambda \varepsilon (0,+\infty)$

$$\underline{S}_\alpha(\lambda) = \frac{||z_\alpha(\lambda)||}{||w_\alpha(\lambda)||} = \frac{||\bar{v}_\alpha||}{||\bar{u}_\alpha||} \cdot \lambda^{\left(\mathbb{B}\left(\frac{v}{||v||}\right)^{-1} - \mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)} .$$

For the positive scalar

$$\underline{\lambda} := \left(\frac{||\bar{u}_\alpha||}{||\bar{v}_\alpha||}\right)^{\left(\mathbb{B}\left(\frac{v}{||v||}\right)^{-1} - \mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)^{-1}}$$

$\underline{S}_\alpha(\lambda) > 1$ for $\lambda > \underline{\lambda}$ and also $\underline{S}_i(\lambda) > 1$ for $\lambda > \underline{\lambda}$ and $i \varepsilon (\{1,2, \ldots, m\} \sim \{\alpha\})$, implying $z_i(\lambda) > w_i(\lambda)$ for $\lambda > \underline{\lambda}$, $i \varepsilon \{1,2, \ldots, m\}$. Consequently, there would exist a positive scalar $\epsilon > 0$ such that for $\lambda > \bar{\lambda}$, $w(\lambda) \varepsilon \text{ ISOQ } \mathbb{P}(\lambda\bar{x})$, $z(\lambda) \text{ ISOQ } \mathbb{P}(\lambda\bar{x})$, $(1 + \epsilon)w(\lambda) \notin \mathbb{P}(\lambda\bar{x})$, but $(1 + \epsilon)w(\lambda) \leq z(\lambda)$ contradicting $\mathbb{P}$.6SS. Hence if $\mathbb{P}$.6SS is to hold, it is necessary that $\mathbb{B}\left(\frac{u}{||u||}\right) \leqq \mathbb{B}\left(\frac{v}{||v||}\right)$.

Next suppose $\mathbb{B}\left(\frac{u}{||u||}\right) < \mathbb{B}\left(\frac{v}{||v||}\right)$. Here let β denote the component identifying subscript of the components of $\bar{u}$ and $\bar{v}$ such that

$$\bar{S}_\beta := \frac{||\bar{v}_\beta||}{||\bar{u}_\beta||} = \operatorname*{Max}_i \left\{\frac{||\bar{v}_i||}{||\bar{u}_i||} : i \varepsilon \{1,2, \ldots, m\}\right\} .$$

Define

$$\bar{S}_i(\lambda) := \frac{||w_i(\lambda)||}{||z_i(\lambda)||} = \frac{||\bar{u}_i||}{||\bar{v}_i||} \lambda^{\left(-\mathbb{B}\left(\frac{v}{||v||}\right)^{-1} + \mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)}$$

$$\bar{\lambda} := \left(\frac{||\bar{v}_\beta||}{||\bar{u}_\beta||}\right)^{\left(-\mathbb{B}\left(\frac{v}{||v||}\right)^{-1} + \mathbb{B}\left(\frac{u}{||u||}\right)^{-1}\right)^{-1}} .$$

Then for $0 < \lambda < \bar{\lambda}$, $\bar{S}_i(\lambda) < 1$ for $i \varepsilon \{1,2, \ldots, m\}$, implying $z(\lambda) < w(\lambda)$ for $0 < \lambda < \bar{\lambda}$. Consequently, there exists a positive scalar $\epsilon > 0$ such that for $0 < \lambda < \bar{\lambda}$, $w(\lambda) \varepsilon$ ISOQ $\mathbb{P}(\lambda\bar{x})$, $z(\lambda) \varepsilon$ ISOQ $\mathbb{P}(\lambda\bar{x})$, $(1 + \epsilon)z(\lambda) \not\in \mathbb{P}(\lambda\bar{x})$, but $(1 + \epsilon)z(\lambda) \leq w(\lambda)$, contradicting the property $\mathbb{P}$.6SS. Thus $\mathbb{B}\left(\frac{u}{||u||}\right) = \mathbb{B}\left(\frac{v}{||v||}\right)$.

Then, assuming that $\mathbb{L}(u) \neq \emptyset$ for some $u \varepsilon (L_\infty)^m_{++}$, $\mathbb{L}(u) \neq \emptyset$ for all $u \varepsilon (L_\infty)^m_{++}$, and by the construction given above one may conclude that $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for all $u \varepsilon (L_\infty)^m_{++}$ when property $\mathbb{P}$.6SS holds for a semi-homogeneous dynamic correspondence $u \to \mathbb{L}(u)$.

This result may be extended to all $u \varepsilon (L_\infty)^m_+$, $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, when it is required that for some $u \varepsilon (L_\infty)^m_{++}$, $\mathbb{L}(u) \neq \emptyset$. Let $v \geq 0$, $v \nmid 0$ be arbitrarily chosen. Then there exists $u > 0$ such such that $v \leq u$. The foregoing argument for $\mathbb{B}\left(\frac{u}{||u||}\right) < \mathbb{B}\left(\frac{v}{||v||}\right)$ may be carried out without change to show that $\mathbb{P}$.6SS implies $\mathbb{B}\left(\frac{u}{||u||}\right) \geqq \mathbb{B}\left(\frac{v}{||v||}\right)$. For the case $\mathbb{B}\left(\frac{u}{||u||}\right) > \mathbb{B}\left(\frac{v}{||v||}\right)$, the argument may be altered by defining:

$$\underline{S}_\alpha := \frac{||\bar{u}_\alpha||}{||\bar{v}_\alpha||} = \underset{i}{\text{Min}} \left\{\frac{||\bar{u}_i||}{||\bar{v}_i||} : i \varepsilon \{1,2, \ldots, m\}\right\}$$

$$\underline{S}_\alpha(\lambda) := \frac{||w_\alpha(\lambda)||}{||z_\alpha(\lambda)||} = \frac{||\bar{u}_\alpha||}{||\bar{v}_\alpha||} \lambda^{\left(\mathbb{B}\left(\frac{u}{||u||}\right)^{-1} - \mathbb{B}\left(\frac{v}{||v||}\right)^{-1}\right)}$$

$$\underline{\lambda} := \frac{||\bar{v}_\alpha||}{||\bar{u}_\alpha||} \cdot \lambda\left(\mathbb{B}\left(\frac{u}{||u||}\right)^{-1} - \mathbb{B}\left(\frac{v}{||v||}\right)^{-1}\right)^{-1} .$$

Then for $\lambda > \underline{\lambda}$, $S_i(\lambda) > 1$ for $i \in \{1,2, \ldots, m\}$, implying $w(\lambda) > z(\lambda)$ for $\lambda > \underline{\lambda}$. Consequently, there exists a positive scalar $\epsilon > 0$ such that for $\lambda > \underline{\lambda}$, $w(\lambda) = \text{ISOQ } \mathbb{P}(\bar{\lambda}x)$, $z(\lambda) \in \text{ISOQ } \mathbb{P}(\bar{\lambda}x)$, $(1 + \epsilon)z(\lambda) \notin \mathbb{P}(\lambda\bar{x})$, but $(1 + \epsilon)z(\lambda) \leq w(\lambda)$ contradicting $\mathbb{P}$.6SS.

In case $\mathbb{L}(u) = \emptyset$ for all $u \in (L_\infty)^m_{++}$, one may show that $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for all

$$u \in \left\{u \in (L_\infty)^m_+ : u_i > 0 \text{ for } i \in \Sigma_k , u_i = 0 \text{ for } i \notin \Sigma_k\right\}$$

where $\Sigma_k = \{\nu_1, \nu_2, \ldots, \nu_k\} \subset \{1,2, \ldots, n\}$, $(1 \leqq k < n)$. But, one cannot show that $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for all vectors $u \in (L_\infty)^m_+$, $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, where the null output histories may be for different outputs. Indeed, the axiom systems of Chapter 2 do not even require that more than one output may have a nonnull output rate history for a possible output vector, to allow for the possibility that the various outputs are pure alternatives. Then for each $u_i \in (L_\infty)_+$, $u_i \geq 0$, $\mathbb{B}\left(\frac{(0,0, \ldots, u_i, 0, \ldots, 0)}{||u||}\right)$ is a positive constant, $i \in \{1,2, \ldots, m\}$.

The following proposition has been shown:

<u>Proposition (5.2-1)</u>: (Shephard, 1974:b)

A dynamic semi-homogeneous correspondence $u \rightarrow \mathbb{L}(u)$ may satisfy property $\mathbb{P}$.6SS $\Longleftrightarrow$ $\mathbb{L}$.6SS if and only if:

(a) $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for $u \geq 0$, if $\mathbb{L}(v) \neq \emptyset$ for some $v >>> 0$, $v \in (L_\infty)^m_+$.

(b) If $w \geq 0$, $w \not> 0$, $\mathbb{L}(w) \neq 0$, $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for $u \in \left\{v \in (L_\infty)^m_+ : v \leqq (\theta_1 w_1, \theta_2 w_2, \ldots, \theta_m w_m) , \theta_i \in (0,+\infty) , i \in \{1,2, \ldots, m\}\right\}$, in case $\mathbb{L}(u) = \emptyset$ for $u \in (L_\infty)^m_{++}$.

Sub-Proposition (5.2-1):

If the dynamic production correspondence $u \to \mathbb{L}(u)$ is Semi-Homogeneous under axioms $\mathbb{P}.6S \iff \mathbb{L}.6S$, and $\mathbb{L}(w) \neq \emptyset$, $w \geq 0$, $\mathbb{B}\left(\frac{u}{||u||}\right)$ is a positive constant for $u \in \left\{v \in (L_\infty)^m_+ : v = (\theta_1 \theta w_1, \theta_2 \theta w_2, \ldots, \theta_m \theta w_m) , \theta \in (0,+\infty) , \theta_i \in (0,1) , i \in \{1,2, \ldots, m\}\right\}$.

Note that by semi-homogeneity, if $\mathbb{L}(u) \neq \emptyset$, $\mathbb{L}(\theta u) \neq \emptyset$ for $\theta \in (0,+\infty)$. *A proof similar to that of* Proposition (5.2-1) can be carried out.

Semi-homogeneous dynamic output correspondences are independently defined by:

Definition (5.2-2): (Eichhorn, 1969 and Shephard, 1974:b)

The dynamic production correspondence $\mathbb{P} : x \in (L_\infty)^n_+ \to \mathbb{P}(x) \in 2^{(L_\infty)^m_+}$ is Semi-Homogeneous if and only if $\mathbb{P}(\lambda x) = \lambda^{\mathbb{D}\left(\frac{x}{||x||}\right)} \mathbb{P}(x)$, $\lambda \in (0,+\infty)$, $\mathbb{P}(x) \neq \{0\}$, where $\mathbb{D} : x \in (L_\infty)^n_+ \to \mathbb{D}\left(\frac{x}{||x||}\right) \in R_+$ is a positive bounded functional for $x \in \left\{x \in (L_\infty)^n_+ : ||x|| = 1\right\}$, constant along rays in $(L_\infty)^n_+$.

As in the case of the semi-homogeneous dynamic correspondence, it holds that:

Lemma (5.2-2):

If the dynamic production correspondence $x \to \mathbb{P}(x)$ is semi-homogeneous, $u \in \text{ISOQ } \mathbb{P}(x) \iff x \in \text{ISOQ } \mathbb{L}(u)$.

Also by arguments paralleling those given in the case of a semi-homogeneous correspondence $u \to \mathbb{L}(u)$, one may show:

Proposition (5.2-2): (Shephard, 1974:b)

A dynamic semi-homogeneous correspondence $x \to \mathbb{P}(x)$ may satisfy property $\mathbb{P}$.6SS if and only if $\mathbb{D}\left(\frac{x}{||x||}\right)$ is a positive constant for $x \in (L_\infty)^n_+$, $\mathbb{P}(x) \neq \{0\}$.

Here the qualifications regarding the existence of $u >>> 0$, $\mathbb{L}(u) \neq \emptyset$ are not required. See axioms $\mathbb{P}.4.2 \iff \mathbb{L}.4.2$, $\mathbb{P}.2 \iff \mathbb{L}.2$ and $\mathbb{P}.4.1$.

Sub-Proposition (5.2-2):

If the dynamic production correspondence $x \to \mathbb{P}(x)$ is Semi-Homogeneous under axioms $\mathbb{L}.3S \iff \mathbb{P}.3S$, $z \in (L_\infty)^n_+$, $\mathbb{P}(z) \neq \{0\}$, $\mathbb{D}\left(\frac{x}{||x||}\right)$ is a positive constant for $x \in \left\{y \in (L_\infty)^n_+ : y = (\lambda_1\lambda z_1, \lambda_2\lambda z_2, \ldots, \lambda_n\lambda z_n) , \lambda \in (0,+\infty) , \lambda_i \in (1,+\infty) , i \in \{1,2, \ldots, n\}\right\}$.

Note that by semi-homogeneity, $\mathbb{P}(\lambda z) \neq \{0\}$ for $\lambda \in (0,+\infty)$, $\mathbb{P}(z) \neq \{0\}$.

Next, suppose both the dynamic production correspondence $x \to \mathbb{P}(x)$ and its inverse correspondence $u \to \mathbb{L}(u)$ are semi-homogeneous. It is to be expected that the exponent functionals $\mathbb{B}\left(\frac{u}{||u||}\right)$ and $\mathbb{D}\left(\frac{x}{||x||}\right)$ would have a special relationship to each other. Let $u \in (L_\infty)^m_+$, $u \geq 0$ and $\mathbb{L}(u) \neq \emptyset$. Then for some $x \in (L_\infty)^n_+$, $x \geq 0$, one has $x \in \mathbb{L}(u)$. By axiom $\mathbb{L}.4.2$, there exists for each $\theta \in (0,+\infty)$ a

positive scalar λ_θ such that $\lambda_\theta \, \varepsilon \, \mathbb{L}(\theta u)$ for any $x \, \varepsilon \, \mathbb{L}(u)$.
Using the semi-homogeneity property,

$$(\lambda_\theta x) \, \varepsilon \, \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u) \iff \theta^{-\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot (\lambda_\theta x) \, \varepsilon \, \mathbb{L}(u) \iff$$

$$u \, \varepsilon \, \mathbb{P}\left(\theta^{-\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \lambda_\theta x\right) = \theta^{-\mathbb{B}\left(\frac{u}{||u||}\right) \cdot \mathbb{D}\left(\frac{x}{||x||}\right)} \cdot \mathbb{P}(\lambda_\theta x) \iff$$

$$\lambda_\theta x \, \varepsilon \, \mathbb{L}\left(\theta^{\mathbb{B}\left(\frac{u}{||u||}\right) \cdot \mathbb{D}\left(\frac{x}{||x||}\right)} \cdot u\right) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)^2 \cdot \mathbb{D}\left(\frac{x}{||x||}\right)} \cdot \mathbb{L}(u) .$$

Hence

$$\theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)^2 \cdot \mathbb{D}\left(\frac{x}{||x||}\right)} \cdot \mathbb{L}(u) .$$

Therefore

$$\mathbb{D}\left(\frac{x}{||x||}\right) = \frac{1}{\mathbb{B}\left(\frac{u}{||u||}\right)}$$

for $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$ and $x \, \varepsilon \, \mathbb{L}(u)$.

Thus the following proposition holds:

Proposition (5.2-3): (Shephard, 1974:b)

If the dynamic production correspondence $x \rightarrow \mathbb{P}(x)$ and its inverse correspondence $u \rightarrow \mathbb{L}(u)$ are both semi-homogeneous, the exponent functional $\mathbb{D}\left(\frac{x}{||x||}\right)$ is a positive constant equal to $\mathbb{B}\left(\frac{u}{||u||}\right)^{-1}$ for all $x \, \varepsilon \, \mathbb{L}(u)$.

These two propositions prompt the terminology semi-homogeneity used in definitions (5.2-1) and (5.2-2). Connectivity of the map sets $\mathbb{L}(u)$ is not required at all by the axiom system of Chapter 2. Indeed the dynamic model of production so formulated accommodates the case where the map sets $\mathbb{L}(u)$ consist only of sections of rays in the space $(L_\infty)_+^n$, finite number perhaps, to describe discrete factor-time-distribution mix for input rate histories, resulting possibly in only a finite number of ray segments in $(L_\infty)_+^m$ as output-time-distribution, consistent with Proposition (5.2-3).

As an example of a semi-homogeneous dynamic output correspondence, modify relation (2.3-20) to

$$\mathbb{P}(x) := \begin{cases} \left\{ u \in (L_\infty)_+^m : u_\nu \leqq \left[\sum_{j=1}^{n} \beta_{j\nu} (X_{j\nu})^{-\rho} \right]^{\frac{-\mathbb{D}\left(\frac{x}{\|x\|}\right)}{\rho}} ,\ \nu = 1, 2, \ldots, (N+1) \right\}, & x \in X \\ \{0\} , & x \notin X . \end{cases}$$

Then

$$\mathbb{P}(\lambda x) = \lambda^{\mathbb{D}\left(\frac{x}{\|x\|}\right)} \cdot \mathbb{P}(x) , \quad x \in (L_\infty)_+^n ,$$

and the dynamic output correspondence so defined is semi-homogeneous. For the dynamic input correspondence, consider

$$\mathbb{L}(u) := \left\{ x \in X : \left[\sum_{1}^{n} \beta_{j\nu}(X_{j\nu})^{-\rho} \right]^{\frac{-\mathbb{B}\left(\frac{u}{||u||}\right)^{-1}}{\rho}} \geqq u_{\nu} \ , \ \nu = 1,2, \ldots, (N+1) \right\} .$$

Here,

$$\mathbb{L}(\theta u) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u)$$

and $\mathbb{L}(u)$ so defined is semi-homogeneous. Note that, if both apply for the same system, $\mathbb{B}\left(\frac{u}{||u||}\right)^{-1} = \mathbb{D}\left(\frac{x}{||x||}\right)$ for $x \in X$.

5.3 Ray Homothetic Correspondences

The definitions of Section 5.1 for dynamic homothetic production correspondences, state that all output (input) map sets may be scaled in terms of a fixed output (input) set. A less restrictive scaling is one where for given input (output) mix a scaling of the given mix results in a scaling of output (input) set in terms of a fixed output (input) set depending upon the input (output) mix being scaled. Precisely, this scaling is defined by:

Definition (5.3-1): (Färe and Shephard, 1977)

A dynamic production correspondence $\mathbb{P} : x \in (L_\infty)_+^n \to \mathbb{P}(x) \in 2^{(L_\infty)_+^m}$ is Ray Homothetic if and only if

$$(5.3\text{-}1) \quad \mathbb{P}(x) = \frac{F(\mathbb{H}(x))}{F\left(\mathbb{H}\left(\frac{x}{||x||}\right)\right)} \cdot \mathbb{P}\left(\frac{x}{||x||}\right) \ , \quad \mathbb{P}\left(\frac{x}{||x||}\right) \neq \{0\}$$

under $F(\cdot)$, $\mathbb{H}(x)$ are as defined in Section 5.1.

<u>Definition (5.3-2):</u> (Färe and Shephard, 1977)

A dynamic production correspondence $\mathbb{L} : u \in (L_\infty)_+^m \to \mathbb{L}(u) \in 2^{(L_\infty)_+^n}$ is Ray Homothetic if and only if

$$\mathbb{L}(u) = \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{||u||}\right)\right)} \cdot \mathbb{L}\left(\frac{u}{||u||}\right) , \quad \mathbb{L}\left(\frac{u}{||u||}\right) \neq \emptyset \tag{5.3-2}$$

where $G(\cdot)$, $\mathbb{J}(u)$ are as defined in Section 5.1.

The global homotheticity of production correspondence defined in Section 5.1 is merely a special case of ray homotheticity. For example, if $\mathbb{P}(x) = \mathbb{F}(\mathbb{H}(x)) \cdot \mathbb{P}_{\mathbb{ff}}(1)$ then

$$\mathbb{P}\left(\frac{x}{||x||}\right) = F\left(\mathbb{H}\left(\frac{x}{||x||}\right)\right) \cdot \mathbb{P}_{\mathbb{ff}}(1)$$

and Equation (5.3-1) follows by substituting for $\mathbb{P}_{\mathbb{ff}}(1)$.

Semi-homogeneity is likewise a special case of ray homotheticity. For example, if $F(\cdot)$ is taken as the Identity Function and $\mathbb{H}(x)$ satisfies

$$\mathbb{H}(\lambda x) = \lambda^{\mathbb{D}\left(\frac{x}{||x||}\right)} \cdot \mathbb{H}(x) , \tag{5.3-3}$$

then

$$\begin{aligned} \mathbb{P}(\lambda x) &= \frac{F(\mathbb{H}(\lambda x))}{F\left(\mathbb{H}\left(\frac{x}{||x||}\right)\right)} \cdot \mathbb{P}\left(\frac{x}{||x||}\right) = \frac{F(\mathbb{H}(\lambda x))}{F(\mathbb{H}(x))} \cdot \mathbb{P}(x) \\ &= \lambda^{\mathbb{D}\left(\frac{x}{||x||}\right)} \cdot \mathbb{P}(x) . \end{aligned}$$

Evidently (5.3-3) does not conflict with the properties taken for $F(\cdot)$ and $\mathbb{H}(x)$. In a similar way it can be seen that semi-homogeneity of $u \to \mathbb{L}(u)$ is a special case of ray homotheticity for $u \to \mathbb{L}(u)$.

Later, when cost functionals and expansion paths are considered, ray homotheticity will play an important role. Meanwhile it is interesting to consider the implications of ray homotheticity holding for both the output correspondence $x \to \mathbb{P}(x)$ and the inverse correspondence $u \to \mathbb{L}(u)$ of the same production structure.

If input and output dynamic correspondences are both ray homothetic,

$$(5.3\text{-}4) \qquad \mathbb{P}(\lambda x) = \frac{F(\mathbb{H}(\lambda x))}{F(\mathbb{H}(x))} \cdot \mathbb{P}(x) \ , \ \mathbb{P}(x) \neq \{0\} \ , \ \lambda \in (0,+\infty) \ ,$$

$$(5.3\text{-}5) \qquad \mathbb{L}(\theta u) = \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot \mathbb{L}(u) \ , \ \mathbb{L}(u) \neq \emptyset \ , \ \theta \in (0,+\infty) \ .$$

These relations imply

$$(5.3\text{-}6) \qquad \mathbb{P}(\lambda x) = \Delta(\lambda,x) \cdot \mathbb{P}(x) \ , \ \lambda \in (0,+\infty) \ , \ \mathbb{P}(x) \neq \{0\} \ ,$$

$$(5.3\text{-}7) \qquad \mathbb{L}(\theta u) = \nabla(\theta,u) \cdot \mathbb{L}(u) \ , \ \theta \in (0,+\infty) \ , \ \mathbb{L}(u) \neq \emptyset \ ,$$

where Δ and ∇ are mappings

$$(5.3\text{-}8) \qquad \Delta : R_{++} \times (L_\infty)^n_+ \to R_{++} \ , \ \Delta(1,x) = \Delta(\lambda,0) = 1 \ ,$$

$$(5.3\text{-}9) \qquad \nabla : R_{++} \times (L_\infty)^m_+ \to R_{++} \ , \ \nabla(1,u) = \nabla(\theta,0) = 1 \ ,$$

strictly increasing in λ and θ respectively.

Now assuming that (5.3-4) and (5.3-5) refer to inversely related correspondences, let $x \in (L_\infty)^n_+$ and $u \in (L_\infty)^m_+$, $u \geq 0$, $x \geq 0$,

be a feasible pair of vectors of input and output rate histories. Then, by axiom $\mathbb{L}.4.2$, it follows that there exists a positive scalar λ_θ for each $\theta \in (0,+\infty)$ such that $\lambda_\theta x \in \mathbb{L}(\theta u)$. Then, using (5.3-6) and (5.3-7),

$$\lambda_\theta x \in \mathbb{W}(\theta,u) \cdot \mathbb{L}(u) \iff \frac{\lambda_\theta x}{\mathbb{W}(\theta,u)} \in \mathbb{L}(u) \iff$$

$$u \in \mathbb{P}\left(\frac{\lambda_\theta x}{\mathbb{W}(\theta,u)}\right) = \Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right) \cdot \mathbb{P}(\lambda_\theta) \iff$$

$$\frac{u}{\Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right)} \in \mathbb{P}(\lambda_\theta x) \iff$$

$$\lambda_\theta x \in \mathbb{L}\left(\frac{u}{\Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right)}\right) = \mathbb{W}\left(\frac{1}{\Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right)}, u\right) \cdot \mathbb{L}(u) .$$

Thus

$$\mathbb{W}\left(\frac{1}{\Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right)}, u\right) = \mathbb{W}(\theta,u)$$

and

$$\frac{1}{\theta} = \Delta\left(\frac{1}{\mathbb{W}(\theta,u)}, \lambda_\theta x\right) , \quad \theta \in (0,+\infty) .$$

Thus

(5.3-10) $$\Delta^{-1}\left(\frac{1}{\theta}, \lambda_\theta x\right) \cdot \mathbb{W}(\theta,u) = 1 , \quad \theta \in (0,+\infty) .$$

Similarly, one obtains

(5.3-11) $$\mathbb{W}^{-1}\left(\frac{1}{\theta}, \sigma_\theta u\right) \cdot \Delta(\theta,x) = 1 , \quad \theta \in (0,+\infty)$$

where $(\sigma_\theta u) \in \mathbb{P}(\theta x)$, $\theta \in (0,+\infty)$.

Next, using the axiom $\mathbb{P}.6 \Longleftrightarrow \mathbb{L}.6$, $\mathbb{L}(\mu\theta u) \supset \mathbb{L}(\theta u)$, $\theta \in (0,+\infty)$ and $\mu \in (0,1]$. Then by the previous argument, (5.3-10) becomes

$$\Delta^{-1}\left(\frac{1}{\theta}\ ,\ \lambda_\theta x\right) \cdot \mathbb{W}(\theta,\mu u) = 1\ ,\ \theta \in (0,+\infty)\ ,\ \mu \in (0,1]\ .$$

Accordingly:

$$\mathbb{W}(\theta,u) = \mathbb{W}(\theta,\mu u)\ ,\ \theta \in (0,+\infty)\ ,\ \mu \in (0,1]\ . \tag{5.3-12}$$

For $||u|| \geqq 1$, take $\mu = \frac{1}{||u||}$, and

$$\mathbb{W}(\theta,u) = \mathbb{W}\left(\theta,\ \frac{u}{||u||}\right)\ ,\ \theta \in (0,+\infty)\ ,\ ||u|| \geqq 1\ . \tag{5.3-13}$$

It remains to extend Equation (5.3-13) for $||u|| \leqq 1$. Consider $||u|| \in (0,1]$. Choose μ so that $||\mu u|| \geqq 1$, i.e., $\mu \geqq \frac{1}{||u||} \geqq 1$. Then, by (5.3-13),

$$\mathbb{W}(\theta,\mu u) = \mathbb{W}\left(\theta,\ \frac{u}{||u||}\right)\ ,\ \theta \in (0,+\infty)\ ,\ ||\mu u|| \geqq 1\ .$$

Hence

$$\mathbb{W}(\theta,\mu u) = \mathbb{W}(\theta,u) = \mathbb{W}\left(\theta,\ \frac{u}{||u||}\right)\ , \tag{5.3-14}$$

for $\theta \in (0,+\infty)$ and $\mu \in (0,+\infty)$. From (5.3-7)

$$\begin{aligned}\mathbb{L}(\mu\theta u) &= \mathbb{W}(\mu\theta,u)\mathbb{L}(u)\\ &= \mathbb{W}(\theta,\mu u) \cdot \mathbb{W}(\mu,u)\mathbb{L}(u)\end{aligned}$$

and in the scalar argument, $\mathbb{W}(\theta,u)$ obeys the functional equation

$$\mathbb{W}(\theta\mu,u) = \mathbb{W}(\theta,\mu u) \cdot \mathbb{W}(\mu,u) \ . \tag{5.3-15}$$

Combining (5.3-14) and (5.3-15), one obtains the following Cauchy functional equation satisfied by $\mathbb{W}(\theta,u)$ in the scalar argument:

$$\mathbb{W}\left(\theta\mu, \frac{u}{\|u\|}\right) = \mathbb{W}\left(\theta, \frac{u}{\|u\|}\right) \cdot \mathbb{W}\left(\mu, \frac{u}{\|u\|}\right) \ . \tag{5.3-16}$$

The solution of this equation has been carried out by (Eichhorn, 1969) to show (using 5.3-14) that

$$\mathbb{W}(\theta,u) = \theta^{\mathbb{B}\left(\frac{u}{\|u\|}\right)} \ , \ \theta \in (0,+\infty) \ . \tag{5.3-17}$$

Accordingly, by (5.3-7)

$$\mathbb{L}(\theta u) = \theta^{\mathbb{B}\left(\frac{u}{\|u\|}\right)} \cdot \mathbb{L}(u) \ ,$$

i.e., the correspondence $u \rightarrow \mathbb{L}(u)$ is semi-homogeneous.

By analogous argument, starting with (5.3-11) one obtains

$$\Delta(\theta,x) = \theta^{\mathbb{D}\left(\frac{x}{\|x\|}\right)} \ , \ \theta \in (0,+\infty) \tag{5.3-18}$$

whence

$$\mathbb{P}(\theta x) = \theta^{\mathbb{D}\left(\frac{x}{\|x\|}\right)} \cdot \mathbb{P}(x) \ , \ \theta \in (0,+\infty)$$

and the correspondence $x \rightarrow \mathbb{P}(x)$ is likewise semi-homogeneous. Substitution of (5.3-17) and (5.3-18) into (5.3-10) and (5.3-11), shows that

(5.3-19) $$\mathbb{B}\left(\frac{u}{||u||}\right) = \frac{1}{\mathbb{D}\left(\frac{x}{||x||}\right)} .$$

Thus the following proposition has been shown:

Proposition (5.3-1):

If inversely related dynamic production correspondences $x \to \mathbb{P}(x)$, $u \to \mathbb{L}(u)$ are ray homothetic, they are semi-homogeneous, with reciprocal exponent functions $\mathbb{B}\left(\frac{u}{||u||}\right)$, $\mathbb{D}\left(\frac{x}{||x||}\right)$ which have constant value for all $x \in \mathbb{L}(u)$ for connected map sets $\mathbb{L}(u)$, $\mathbb{L}(u) \neq \emptyset$.

The restriction made that $\mathbb{\Delta}$ and $\mathbb{W}$ are strictly increasing in λ and θ respectively is not essential. See (Eichhorn, 1978).

As examples of ray homothetic dynamic production correspondences, consider

$$\mathbb{P}(x) = \frac{F(\underline{\bar{\Psi}}(1,x))}{F\left(\underline{\bar{\Psi}}\left(1, \frac{x}{||x||}\right)\right)} \cdot \mathbb{P}\left(\frac{x}{||x||}\right)$$

where $\underline{\bar{\Psi}}(1,x)$ is given by (5.1-10) and $\underline{\bar{\Psi}}\left(1, \frac{x}{||x||}\right)$ is obtained from the same expression by substituting $\frac{x}{||x||}$ for x where

$$||x|| = \underset{j}{\mathrm{Max}} \left\{ \underset{\nu}{\mathrm{Max}}\ x_{j\nu} : \nu = 1,2, \ldots, (N+1) , j \in \{1,2, \ldots, n\} \right\} .$$

The set $\mathbb{P}\left(\frac{x}{||x||}\right)$, fixed under scalar extensions of x , is defined by

$$\mathbb{P}\left(\frac{x}{||x||}\right) = \left\{ u \in (L_\infty)^m_+ : \Omega\left(\frac{x}{||x||}, u\right) \leqq 1 \right\}$$

in which $\Omega\left(\frac{x}{||x||}, u\right)$ is given by

$$\Omega\left(\frac{x}{||x||}, u\right) = \left[\underset{\nu}{\mathrm{Min}}\left(\mathrm{Max}\left\{\theta_\nu : \theta_\nu u_\nu \leq \left[\sum_{j=1}^{n} \beta_{j\nu}\left(\frac{X_{j\nu}}{||x||}\right)^{-\rho}\right]^{-\frac{1}{\rho}}\right\}\right)\right]^{-1}$$

where $\nu \in \{1,2, \ldots, (N+1)\}$ and $X_{j\nu}$ is defined by (2.3-16).

CHAPTER 6

COST AND REVENUE FUNCTIONALS

In the previous chapters efficiency has been considered only in a technical sense, where no input rate history may be decreased over a subset of $[0,+\infty)$ of positive measure, the other input rate histories held fixed, and still attain a given vector u of output rate histories. Input vectors $x \in \mathbb{L}(u)$ with this property form the efficient subset $\mathbb{E}(u)$ of $\mathbb{L}(u)$. The many alternatives of $\mathbb{E}(u)$ are all equally efficient in a technical sense. From an economic viewpoint a vector $x \in \mathbb{E}(u)$ is economically efficient under price histories $p \in (L_1)_+^n$, $p_i \in (L_1)_+$ applying to $x_i \in (L_\infty)_+$, if the cost of x equals the minimal cost under p of attaining u.

Similarly those vectors $u \in (L_\infty)_+^m$ obtainable from $x \in (L_\infty)_+^n$ are economically efficient under a price structure $r \in (L_1)^m$ if the revenue obtained from them equals the maximal value possible under $r \in (L_1)^m$.

Thus we are led to consider minimal cost and maximal revenue functionals. For the expansion of output vector u (input vector x), the resulting sets of economically efficient input (output) vectors will be of interest, particularly those with linear structure.

6.1 Minimal Cost Functional

Let $p \in (L_1)_+^n$ denote a vector of summable price histories $p_i \in (L_1)_+$ for $x_i \in (L_\infty)_+$ respectively, $i \in \{1,2, \ldots, n\}$, and consider a vector $u \in (L_\infty)_+^m$ of output histories. The summability of p_i may be interpreted either as being positive only over a finite horizon, or as discounting of value in a compound way to measure the

current value for an indefinite future stream of input. A Minimal Cost Functional is defined for $u \varepsilon (L_\infty)^m_+$, $p \varepsilon (L_1)^m_+$ by

$$(6.1\text{-}1) \qquad \mathbb{K}(u,p) := \underset{x}{\text{Min}} \{\langle p,x\rangle : x \,\varepsilon\, \mathbb{L}(u)\}$$

where

$$(6.1\text{-}2) \qquad \langle p,x\rangle := \sum_{i=1}^{n}\left(\int_0^\infty p_i(t)x_i(t)d\mu_i(t)\right) .$$

For $\mathbb{L}(u) \neq \emptyset$, $\mathbb{L}(u) \subset \text{CLOSURE } \mathbb{E}(u) + (L_\infty)^n_+$, see Proposition (2.2.4-2), and if axiom $\mathbb{E}$.S is taken to apply under the norm topology for $(L_\infty)^n_+$, $\mathbb{K}(u,p)$ exists and is finite, since the minimal value of $\langle p,x\rangle$ occurs on (CLOSURE $\mathbb{E}(u)$) which is compact under $\mathbb{E}$.S. If only the axiom $\mathbb{E}$ is invoked, (CLOSURE $\mathbb{E}(u)$) is compact under the weak* topology for $(L_\infty)^n_+$.

If the given vector u is composed of summable output histories, i.e., $u \varepsilon (\tilde{L}_\infty)^m_+$, and $\bar{t}_u < +\infty$, then $\mathbb{E}(u) = \tilde{\mathbb{E}}(u)$ (see (2.2.3-2), (2.2.3-3) and axiom $\mathbb{L}$.T.2), and one need only invoke the property $\tilde{\mathbb{E}}$.S under the norm topology, or $\tilde{\mathbb{E}}$ under the weak* topology, for $(L_\infty)^n_+$.

When $\mathbb{L}(u) = \emptyset$, clearly $\mathbb{K}(u,p) = +\infty$ for $p \varepsilon (L_1)^m_+$, and, if $u = 0$ or $p = 0$, $\mathbb{K}(u,p) = 0$. The cost functional $\mathbb{K}(u,p)$ is finite for $\mathbb{L}(u) \neq \emptyset$, since

$$\int_0^\infty p_i(t)x_i(t)d\mu_i(t) \ , \ i = 1,2, \ldots, n \ ,$$

is finite for $p_i(t)$ summable while $x_i \varepsilon (L_\infty)_+$, and $\mathbb{K}(u,p) > 0$ for $u \neq 0$, $p >\!>\!> 0$. It is linear homogeneous in $p \varepsilon (L_1)^n_+$, since $\langle p,x\rangle$ is linear homogeneous in p. Moreover

$$\mathbb{K}(u,p+q) \geqq \mathbb{K}(u,p) + \mathbb{K}(u,q) \ ,$$

i.e., the cost functional is superadditive in p . Accordingly $\mathbb{K}(u,p)$ is a concave functional of p and continuous in $p \in (L_1)_+^n$. Depending upon the axiom used for disposability of outputs, the cost functional may be nondecreasing in $u \in (L_\infty)_+^m$, output history homogeneously nondecreasing in u , or homogeneously nondecreasing in u .

The foregoing properties are summarized by the following proposition:

<u>Proposition (6.1-1)</u>:

The minimal cost functional $\mathbb{K}(u,p)$ satisfies:

K.1 $\mathbb{K}(0,p) = \mathbb{K}(u,0) = 0$, $\mathbb{L}(u) \neq \emptyset$, and $\mathbb{K}(u,p) = +\infty$, $p \in (L_1)_+^n$ and $\mathbb{L}(u) = \emptyset$.

K.2 $\mathbb{K}(u,p) \geqq 0$ finite for $\mathbb{L}(u) \neq \emptyset$, and positive for $p >>> 0$.

K.3 $\mathbb{K}(u,\lambda p) = \lambda\, \mathbb{K}(u,p)$, $\lambda \in (0,+\infty)$, $p \in (L_1)_+^n$, $u \in (L_\infty)_+^m$.

K.4 $\mathbb{K}(u,p+q) \geqq \mathbb{K}(u,p) + \mathbb{K}(u,q)$, $u \in (L_\infty)_+^m$, $p \in (L_1)_+^n$.

K.5 $\mathbb{K}(u,p') \geqq \mathbb{K}(u,p)$, $p' \geqq p$, $p' \in (L_1)_+^n$, $u \in (L_\infty)_+^m$.

K.6 $\mathbb{K}(u,p)$ is a concave functional of $p \in (L_1)_+^n$, $u \in (L_\infty)_+^m$.

K.7 $\mathbb{K}(u,p)$ is a continuous functional of $p \in (L_1)_+^n$, $u \in (L_\infty)_+^m$.

K.8 $\mathbb{K}(\theta u,p) \geqq \mathbb{K}(u,p)$, $\theta \in [1,+\infty)$, if $\mathbb{P}$.6 applies, $u \in (L_\infty)_+^m$, $p \in (L_1)_+^n$.

K.8S $\mathbb{K}(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m, p) \geqq \mathbb{K}(u,p)$, $\theta_i \in [1,+\infty)$, $i \in \{1,2,\ldots,m\}$, if $\mathbb{P}$.6S applies, $u \in (L_\infty)_+^m$, $p \in (L_1)_+^n$.

K.8SS $\mathbb{K}(u',p) \geqq \mathbb{K}(u,p)$, $u' \geqq u$, if $\mathbb{P}$.6SS applies, $u \in (L_\infty)_+^m$, $p \in (L_1)_+^n$.

The following proposition is of special interest.

Proposition (6.1-2): (Shephard, 1970:a)

If $u \to \mathbb{L}(u)$ is globally homothetic

$$\mathbb{K}(u,p) = G(\mathbb{J}(u)) \cdot \mathbb{P}(p) \quad \text{for} \quad u \in (L_\infty)^m_+ \, , \; p \in (L_1)^n_+ \, . \tag{6.1-3}$$

By using the definition of global homotheticity,

$$\begin{aligned} \mathbb{K}(u,p) &= \operatorname*{Min}_x \{\langle p,x\rangle : x \in G(\mathbb{J}(u)) \cdot \mathbb{L}_\Phi(1)\} \\ &= G(\mathbb{J}(u)) \cdot \operatorname*{Min}_x \{\langle p,x\rangle : x \in \mathbb{L}_\Phi(1)\} \\ &= G(\mathbb{J}(u)) \cdot \mathbb{M}(p) \end{aligned}$$

where $\mathbb{M}(p)$ is a linear homogeneous functional with the properties implied by those of Proposition (6.1-1). Since $\mathbb{J}(0) = 0$ and $J(u) = +\infty$ for $\mathbb{L}(u) = \emptyset$, Equation (6.1-3) holds for all $p \in (L_1)^n_+$, $u \in (L_\infty)^m_+$. Thus in the case of global homotheticity, the cost functional factors into a homogeneous functional of factor price histories alone, and a functional of the vector of output histories being costed. In this case the scaling functional $G(\mathbb{J}(u))$ for the globally homothetic input correspondence $u \to \mathbb{L}(u)$ is obtainable as "price deflated" minimal cost of u, i.e.,

$$G(\mathbb{J}(u)) = \frac{\mathbb{K}(u,p)}{\mathbb{M}(p)} \, . \tag{6.1-4}$$

One may treat $\mathbb{M}(p)$ as a price level functional for factor prices, since it is linear homogeneous and the following proposition holds:

<u>Proposition (6.1-3)</u>: (Shephard, 1970:a)

$\mathbb{K}(u,p) = \Phi(x^*) \cdot \mathbb{M}(p)$, $u \neq 0$, $\mathbb{L}(u) \neq \emptyset$, where $\Phi(x)$ is the linear homogeneous distance functional for $\mathbb{L}_{\Phi}(1)$, and x^* is a cost minimizing vector x under p .

By using $\Phi(x)$ as a distance functional for $\mathbb{L}_{\Phi}(1)$,

$$\mathbb{K}(u,p) = \min_x \{\langle p,x \rangle : \Phi(x) \geqq G(\mathbb{J}(u))\}$$

and clearly the minimal value of $\langle p,x \rangle$ occurs for $\Phi(x^*) = G(\mathbb{J}(u))$, and

$$\text{(6.1-5)} \qquad \mathbb{K}(u,p) = G(\mathbb{J}(u)) \cdot \mathbb{M}(p) = \Phi(x^*) \cdot \mathbb{M}(p) \ .$$

Thus for cost minimizing vectors $x^* \in (L_\infty)^n_+$, minimal cost is a product of two linear homogeneous functionals one for x^* and the other for vectors p of price histories, which yields a true value of minimal cost. As will be seen later in this chapter when expansion paths are considered, when the output vector u is scaled, the cost minimizing vector x^* is correspondingly scaled to preserve the mix of x^* .

The related form of the distance functional for globally homothetic input correspondence $u \to \mathbb{L}(u)$ is stated in the following proposition:

<u>Proposition (6.1-4)</u>: (Shephard, 1970:a)

If $u \to \mathbb{L}(u)$ is globally homothetic

$$\text{(6.1-6)} \qquad \Psi(u,x) = \frac{\Phi(x)}{G(\mathbb{J}(u))} \ ,$$

where $\Phi(x)$ is the distance functional for $\mathbb{L}_{\Phi}(1)$.

Since the cost minimizing vector x^* satisfies $\bar{\Psi}(u,x^*) = 1$, one may use (6.1-6) to obtain (6.1-5).

In the case of semi-homogeneous input structure, and $\mathbb{L}(u) \neq \emptyset$, $\theta \in (0,+\infty)$,

$$\mathbb{K}(\theta u,p) = \underset{x}{\mathrm{Min}} \left\{ \langle p,x \rangle : x \in \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u) \right\}$$

$$\mathbb{K}(\theta u,p) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \underset{x}{\mathrm{Min}} \{\langle p,x \rangle : x \in \mathbb{L}(u)\}$$

$$\mathbb{K}(\theta u,p) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{K}(u,p) .$$

Hence the cost functional is Semi-Homogeneous and scaling of a vector u of output histories results in homogeneous scaling of the minimal cost of u , of constant degree $\mathbb{B}\left(\frac{u}{||u||}\right)$.

Similarly, one detains ($\mathbb{L}(u) \neq \emptyset$)

$$\begin{aligned} \bar{\Psi}(\theta u,x) &= \left[\mathrm{Min} \left\{ \lambda : \lambda x \in \mathbb{L}(\theta u) = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}(u) \text{ , } \lambda \in [0,+\infty) \right\} \right]^{-1} \\ &= \theta^{-\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \bar{\Psi}(u,x) . \end{aligned}$$

Hence the following proposition holds:

<u>Proposition (6.1-5)</u>: (Shephard, 1974:b)

If the input correspondence $u \to \mathbb{L}(u)$ is semi-homogeneous, minimal cost and distance functional satisfy:

$$(6.1\text{-}7) \qquad \mathbb{K}(\theta u,p) = \theta^{\mathbb{B}\left(\frac{u}{\|u\|}\right)} \cdot \mathbb{K}(u,p)$$

$$(6.1\text{-}8) \qquad \underline{\bar{\Psi}}(\theta u,x) = \theta^{-\mathbb{B}\left(\frac{u}{\|u\|}\right)} \cdot \underline{\bar{\Psi}}(u,x)$$

for $\mathbb{L}(u) \neq \emptyset$, $u \in (L_\infty)^m_+$, $p \in (L_1)^n_+$.

6.2 Maximal Revenue Functional

Let $r \in (L_1)^m$ denote a vector of summable price histories $r_i \in L_1$ for $u_i \in (L_\infty)_+$, respectively, $i \in \{1,2, \ldots, m\}$, and consider a vector $x \in (L_\infty)^n_+$ of input histories. A possibility of negative prices is allowed, because some outputs may be unwanted. As in the case of the cost functional, the summability of r_i may be interpreted either as being nonzero only over a bounded horizon or as a discounting of value in a compound way to measure the current value of a future indefinite stream of output. Then a Maximal Revenue Functional is defined for $x \in (L_\infty)^n_+$, $r \in (L_1)^m$, by

$$(6.2\text{-}1) \qquad \mathbb{R}(x,r) := \operatorname*{Max}_{u} \{\langle r,u \rangle : u \in \mathbb{P}(x)\}$$

where

$$(6.2\text{-}2) \qquad \langle r,u \rangle := \sum_{i=1}^{m} \left(\int_0^\infty r_i(t) u_i(t) d\nu_i(t) \right) .$$

If the axiom $\mathbb{P}.2S$ is invoked, $\mathbb{P}(x)$ is compact under the norm topology, otherwise, if only $\mathbb{P}.2$ is used, the weak* topology for $(L_\infty)^m_+$ is used to obtain $\mathbb{P}(x)$ compact.

When $x = 0$, $\mathbb{P}(x) = \{0\}$ and $\mathbb{R}(x,r) = 0$ for all $r \in (L_1)^m$. Similarly when $r = 0$, $\mathbb{R}(x,r) = 0$ for all $x \in (L_\infty)^n_+$. Clearly $\mathbb{R}(x,r) = 0$ for $r \leq 0$, and $\mathbb{R}(x,r) \geqq 0$ for all $r \in (L_1)^m$. The maximal revenue functional is linear homogeneous in $r \in (L_1)^m$, since $\langle r,u \rangle$ is linear homogeneous. The functional $\mathbb{R}(x,r)$ is obviously subadditive, and nondecreasing in r , since if $r' \geqq r$, $r = r' + \Delta r$ where $r \leqq 0$ and

$$\mathbb{R}(x,r) = \mathbb{R}(x,r' + \Delta r) \leqq \mathbb{R}(x,r') .$$

Hence, $\mathbb{R}(x,r)$ is a convex functional of $r \in (L_1)^m$ and continuous on $(L_1)^m$. Accordingly, as axiom $\mathbb{P}$.3, $\mathbb{P}$.3S or $\mathbb{P}$.3SS is taken to apply, there are three forms for $\mathbb{R}(x,r)$ to be nondecreasing in x .

The foregoing properties are summarized by the following proposition:

<u>Proposition (6.2-1)</u>:

The maximal revenue functional satisfies:

$\mathbb{R}$.1 $\mathbb{R}(0,r) = \mathbb{R}(x,0) = 0$ and $\mathbb{R}(x,r) = 0$ for $r \leq 0$.

$\mathbb{R}$.2 $\mathbb{R}(x,r) \geqq 0$, finite and positive for $r >>> 0$, $\mathbb{P}(x) \neq \{0\}$.

$\mathbb{R}$.3 $\mathbb{R}(x,\theta r) = \theta\, \mathbb{R}(x,r)$, $\theta \in (0,+\infty)$, $r \in (L_1)^m$, $x \in (L_\infty)^n_+$.

$\mathbb{R}$.4 $\mathbb{R}(x,r + s) \leqq \mathbb{R}(x,r) + \mathbb{R}(x,s)$, $x \in (L_\infty)^n_+$, $r \in (L_1)^m$, $s \in (L_1)^m$.

$\mathbb{R}$.5 $\mathbb{R}(x,r') \geqq \mathbb{R}(x,r)$, $r' \geqq r$, $r' \in (L_1)^m$, $r \in (L_1)^m$, $x \in (L_\infty)^n_+$.

$\mathbb{R}$.6 $\mathbb{R}(x,r)$ is a convex functional of $r \in (L_1)^m$, $x \in (L_\infty)^n_+$.

$\mathbb{R}$.7 $\mathbb{R}(x,r)$ is a continuous functional of $r \in (L_1)^m$, $x \in (L_\infty)^n_+$.

$\mathbb{R}.8$ $\quad \mathbb{R}(\lambda x,r) \geqq \mathbb{R}(x,r)$, $\lambda \in [1,+\infty)$, $x \in (L_\infty)^n_+$, $r \in (L_1)^m$, if $\mathbb{P}.3$ applies.

$\mathbb{R}.8S$ $\quad \mathbb{R}(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, r) \geqq \mathbb{R}(x,r)$, $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$, $x \in (L_\infty)^n_+$, $r \in (L_1)^m$, if $\mathbb{P}.3S$ applies.

$\mathbb{R}.8SS$ $\quad \mathbb{R}(x',r) \geqq \mathbb{R}(x,r)$, $x' \geqq x$, $x' \in (L_\infty)^n_+$, $x \in (L_\infty)^n_+$, $r \in (L_1)^m$, if $\mathbb{P}.3SS$ applies.

The following proposition is of special interest.

Proposition (6.2-2): (Shephard, 1970:a)

If $x \to \mathbb{P}(x)$ is globally homothetic

$$\mathbb{R}(x,r) = F(\mathbb{H}(x)) \cdot \mathbb{P}_{ff}(1) \quad \text{for} \quad x \in (L_\infty)^n_+ , \; r \in (L_1)^m . \tag{6.2-3}$$

From the definition of global homotheticity,

$$\begin{aligned} \mathbb{R}(x,r) &= \operatorname*{Max}_u \{\langle r,u\rangle : u \in F(\mathbb{H}(x)) \cdot \mathbb{P}_{ff}(1)\} \\ &= F(\mathbb{H}(x)) \cdot \operatorname*{Max}_u \{\langle r,u\rangle : u \in \mathbb{P}_{ff}(1)\} \\ &= F(\mathbb{H}(x)) \cdot \mathbb{N}(r) \end{aligned}$$

where $\mathbb{N}(r)$ is a linear homogeneous functional with properties implied by those of Proposition (6.2-1). Thus the scaling functional $F(\Phi(x))$ for the globally homothetic output correspondence $x \to \mathbb{P}(x)$ is obtainable as "price deflated" maximal revenue derived from x , i.e.,

$$F(\mathbb{H}(x)) = \frac{\mathbb{R}(x,r)}{\mathbb{N}(r)} . \tag{6.2-4}$$

Here again one may treat $\mathbb{N}(r)$ as a price level functional for output price histories, because of the following proposition.

<u>Proposition (6.2-3)</u>: (Shephard, 1970:a)

$\mathbb{R}(x,r) = \mathbb{ff}(u^*) \cdot \mathbb{N}(r)$, where $\mathbb{ff}(u)$ denotes the linear homogeneous distance functional for $\mathbb{P}_{\mathbb{ff}}(1)$ and u^* is a revenue maximizing vector u under r.

The argument for this proposition is similar to that given for Proposition (6.1-3) and need not be repeated here. Also, without repeating arguments one has:

<u>Proposition (6.2-4)</u>: (Shephard, 1970:a)

If $x \to \mathbb{P}(x)$ is globally homothetic,

$$\Omega(x,u) = \frac{\mathbb{ff}(u)}{F(\mathbb{H}(x))} \tag{6.2-5}$$

where $\mathbb{ff}(u)$ is the distance functional for $\mathbb{P}_{\mathbb{ff}}(1)$.

<u>Proposition (6.2-5)</u>: (Shephard, 1974:b)

If x the output correspondence is semi-homogeneous, the maximal revenue and distance functional satisfy

$$\mathbb{R}(\lambda x,r) = \lambda^{\mathbb{D}\left(\frac{x}{\|x\|}\right)} \cdot \mathbb{R}(x,r)$$

$$\Omega(\lambda x,r) = \lambda^{-\mathbb{D}\left(\frac{x}{\|x\|}\right)} \cdot \Omega(x,r),$$

for $x \in (L_\infty)^n_+$, $r \in (L_1)^m$.

If $x \to \mathbb{P}(x)$ and the corresponding inverse correspondence $u \to \mathbb{L}(u)$ are inversely related globally homothetic, an interesting special result occurs.

Proposition (6.2-6): (Shephard, 1970:a)

If $x \to \mathbb{P}(x)$ and $u \to \mathbb{L}(u) = \left\{x \varepsilon (L_\infty)^n_+ : u \varepsilon \mathbb{P}(x)\right\}$ are inversely related globally homothetic,

$$\frac{\mathbb{R}(x^*,r)}{\mathbb{N}(r)} = F\left(\frac{\mathbb{K}(u^*,p)}{\mathbb{M}(p)}\right) \tag{6.2-6}$$

where $x^* \varepsilon \mathbb{L}(u^*)$, and x^* and u^* are cost minimizing and revenue maximizing vectors respectively for the other.

6.3 Expansion Paths

6.3.1 Scaling of Output Histories

Here we are concerned with how cost minimizing vectors x of input histories will vary if the vector u of output histories to be attained is scaled. Similarly, if a vector x of input histories is scaled, the resulting variation of revenue maximizing vectors u of output histories is of interest.

The following definitions will serve the analysis to follow:

Denote the set of vectors x of $\mathbb{L}(u)$ minimizing cost $\langle p,x \rangle$ of u for $p \geq 0$, $\mathbb{L}(u) \neq \emptyset$, by

$$K_p(u) := \left\{x \varepsilon (L_\infty)^n_+ : x \varepsilon \mathbb{L}(u) , \langle p,x \rangle = \mathbb{K}(u,p)\right\} \tag{6.3.1-1}$$

and let

$$K(u) := \bigcup_{p \geq 0} (K_p(u)) . \qquad (6.3.1\text{-}2)$$

<u>Definition (6.3.1-1)</u>:

The p-Expansion Path Set for an output history mix $\frac{u}{\|u\|}$, $\mathbb{L}(u) \neq \emptyset$, $p \geq 0$, is given by

$$C_p(u) := \bigcup_{\theta \varepsilon [0,+\infty)} (K_p(\theta u)) . \qquad (6.3.1\text{-}3)$$

<u>Definition (6.3.1-2)</u>:

The Global Expansion Path Set for an output history mix $\frac{u}{\|u\|}$, $\mathbb{L}(u) \neq \emptyset$, is given by

$$C(u) := \bigcup_{p \geq 0} (C_p(u)) = \bigcup_{\theta \varepsilon [0,+\infty)} K(\theta u) . \qquad (6.3.1\text{-}4)$$

<u>Definition (6.3.1-3)</u>:

The p-Expansion Path Set $C_p(u)$ has Linear Structure if and only if there exists a scalar $\lambda\left(p,\theta, \frac{u}{\|u\|}\right) \geqq 0$ such that

$$K_p(\theta u) = \lambda(p,\theta,u) \cdot K_p(u) \qquad (6.3.1\text{-}5)$$

for $\theta \; \varepsilon \; [0,+\infty)$.

<u>Definition (6.3.1-4)</u>:

The Global Expansion Path Set $C(u)$ has linear structure if and only if there exists a scalar $\lambda(\theta,u)$ such that

$$K(\theta u) = \lambda(\theta,u) \cdot K(u) \tag{6.3.1-6}$$

for $\theta \varepsilon [0,+\infty)$.

The special structures of Chapter 5 give rise to linear expansion paths. Note that in the case of global homotheticity,

$$\mathbb{L}(u) = G(\mathbb{J}(u)) \cdot \mathbb{L}_{\Phi}(1)$$

$$\mathbb{L}(\theta u) = G(\mathbb{J}(\theta u)) \cdot \mathbb{L}_{\Phi}(1) \ , \ \mathbb{L}(u) \neq \emptyset \ ,$$

and

$$\mathbb{L}(\theta u) = \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot \mathbb{L}(u) \ , \ \mathbb{L}(u) \neq \emptyset \ , \ \theta \ \varepsilon \ (0,+\infty) \ , \tag{6.3.1-7}$$

which is a scaling law for globally homothetic input structures. (See Shephard, 1974:b, p. 277).

Similarly, in the case of semi-homogeneous input structures:

$$\mathbb{L}(u) = (||u||)^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}\left(\frac{u}{||u||}\right)$$

$$\mathbb{L}(\theta u) = (||\theta u||)^{\mathbb{B}\left(\frac{u}{||u||}\right)} \cdot \mathbb{L}\left(\frac{u}{||u||}\right) \ , \ \mathbb{L}(u) \neq \emptyset \ , \ u \neq 0 \ , \ \theta \ \varepsilon \ (0,+\infty)$$

and

$$\mathbb{L}(\theta u) = \frac{(||\theta u||)^{\mathbb{B}\left(\frac{u}{||u||}\right)}}{(||u||)^{\mathbb{B}\left(\frac{u}{||u||}\right)}} \cdot \mathbb{L}(u) \ , \ \mathbb{L}(u) \neq \emptyset \ , \ \theta \ \varepsilon \ (0,+\infty)$$

which is a scaling law of the same form as (6.3.1-7). Further, in the case of ray homothetic input structures

$$\mathbb{L}(u) = G(\mathbb{J}(u)) \cdot \mathbb{L}\left(\frac{u}{||u||}\right)$$

$$\mathbb{L}(\theta u) = G(\mathbb{J}(\theta u)) \cdot \mathbb{L}\left(\frac{u}{||u||}\right) , \quad \mathbb{L}(u) \neq \emptyset , \quad \theta \in (0,+\infty)$$

and again a scaling law of the form (6.3.1-7) holds.

These results lead one to consider input structures for which a law of the form (6.3.1-7) holds. In the case of semi-homogeneous input structures one need only take

(6.3.1-8) $$G(\mathbb{J}(u)) := (\mathbb{J}(u))^{\mathbb{B}\left(\frac{u}{||u||}\right)} .$$

(6.3.1-9) $$\mathbb{J}(u) := ||u|| .$$

Proposition (6.3.1-1): (Färe and Shephard, 1977)

If the input structure $u \to \mathbb{L}(u)$ satisfies the scaling law (6.3.1-7), for $u \in (L_\infty)^m_+$, $\mathbb{L}(u) \neq \emptyset$ it has linear structure for the global expansion path sets $C(u)$.

Clearly $u = 0$ is a degenerate case and need not be considered. Thus let $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$ and $\theta \in [0,+\infty)$. Then $\mathbb{L}(\theta u) \neq \emptyset$. For $p \geq 0$, the minimal cost functional is

$$K(\theta u,p) = \underset{x}{\mathrm{Min}} \left\{ \langle p \cdot x \rangle : x \in \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot \mathbb{L}(u) \right\}$$

$$= \frac{G((\theta u))}{G((u))} \cdot \mathbb{K}(u,p) ,$$

and

$$K_p(\theta u) = \left\{ x \in (L_\infty)^m_+ : x \in \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot \mathbb{L}(u) \ , \ \langle p,x \rangle = \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot \mathbb{L}(u) \right\}$$

$$= \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \cdot K_p(u) \ .$$

Then

$$K(\theta u) = \bigcup_{p \geq 0} K_p(u) = \lambda(\theta,u) \cdot K(u)$$

where

(6.3.1-10) $$\lambda(\theta,u) := \frac{G(\mathbb{J}(\theta u))}{G(\mathbb{J}(u))} \ .$$

In the case of semi-homogeneous input structure, the scalar $\lambda(\theta,u)$ takes the form:

(6.3.1-11) $$\lambda(\theta,u) := \frac{||\theta u||^{\mathbb{B}\left(\frac{u}{||u||}\right)}}{||u||^{\mathbb{B}\left(\frac{u}{||u||}\right)}} = \theta^{\mathbb{B}\left(\frac{u}{||u||}\right)} \ .$$

Without, strong assumptions on the structure of the input correspondence, linear structure of the global expansion path set does not imply the scaling law (6.3.1-7). Thus, linear structure for global expansion path sets is a more general property for the input structure than ray homotheticity and the special forms of this property exhibited by semi-homogeneous, globally homothetic and homogeneous correspondences.

A proposition in this connection is stated without proof:

<u>Proposition (6.3.1-2)</u>: (Färe and Shephard, 1977)

If the input structure $u \to \mathbb{L}(u)$ has convex map sets $\mathbb{L}(u)$, $u \in (L_\infty)^m_+$, and axiom $\mathbb{L}$.3SS applies, linear structure of global expansion

path sets $C(u)$ implies a scaling law of the form (6.3.1-7) in the weak* topology for $u \to \mathbb{L}(u)$.

It is convenient to consider here the form of the minimum cost functional for ray homothetic input structures

$$\mathbb{K}(u,p) = \underset{x}{\text{Min}}\left\{\langle p,x\rangle : x \in \mathbb{L}(u) = \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)} \cdot \mathbb{L}\left(\frac{u}{\|u\|}\right)\right\},$$

(6.3.1-12)

$$\mathbb{K}(u,p) = \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)} \cdot \mathbb{K}\left(\frac{u}{\|u\|},p\right) .$$

Thus, for *given output mix* $\left(\frac{u}{\|u\|}\right)$ one may regard

$$\mathbb{K}\left(\frac{u}{\|u\|},p\right)$$

as a linear homogeneous price level functional, like $\mathbb{P}(p)$ in (6.1-3), and obtain again, for scaling a vector u of output histories, that the scaling functional, here represented by

$$\frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)}, \tag{6.3.1-13}$$

is obtainable as "price deflated" minimal cost of obtaining u , where the price functional used as deflator depends upon the mix $\frac{u}{\|u\|}$ of output histories being scaled. Since

$$\begin{aligned}\mathbb{K}(\theta u,p) &= \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)} \cdot \left\langle p,x^*\left(\frac{u}{\|u\|}\right)\right\rangle \\ &= \left\langle p \cdot \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)},x^*\left(\frac{u}{\|u\|}\right)\right\rangle\end{aligned}$$

it is clear that as the vector u is scaled the cost minimizing vector $x^*\left(\frac{u}{||u||}\right)$ is scaled by the same functional (6.3.1-11), and the mix of input histories of $x^*\left(\frac{u}{||u||}\right)$ is not altered.

6.3.2 Scaling of Input Histories

The definitions for expansion path sets of output histories obtained from scaling of a vector x of input histories follow closely those for expansion path sets of input histories.

Denote the set of vectors u of $\mathbb{P}(x)$ maximizing revenue $\langle r,u\rangle$ for $x \varepsilon \left\{x \varepsilon (L_\infty)^n_+ : \mathbb{P}(x) \neq \{0\}\right\}$ and $r \varepsilon \left\{r \varepsilon (L_1)^m : \mathbb{R}(x,r) > 0\right\} : = (\hat{L}_1)^m$ by

$$(6.3.2\text{-}1) \qquad B_r(x) : = \left\{u \varepsilon (L_\infty)^m_+ : u \varepsilon \mathbb{P}(x) , \langle r,u\rangle = \mathbb{R}(x,r)\right\} ,$$

let

$$(6.3.2\text{-}2) \qquad B(x) : = \bigcup_{r\varepsilon(\hat{L}_1)^m} (B_r(x)) .$$

Expansion path sets and linear structure of the same are then defined by:

Definition (6.3.2-1):

The r-Expansion Path Set for an input mix $\frac{x}{||x||}$, $\mathbb{P}(x) \neq \{0\}$, $r \varepsilon (\hat{L}_1)^m$, is given by

$$(6.3.2\text{-}3) \qquad D_r(x) : = \bigcup_{\lambda\varepsilon[0,+\infty)} (B_r(\lambda x)) .$$

Definition (6.3.2-2):

The Global Expansion Path Set for an input mix $\frac{x}{||x||}$, $\mathbb{P}(x) \neq \{0\}$, $r \in (\hat{L}_1)^m$, is given by

$$(6.3.2\text{-}4) \qquad D(x) := \bigcup_{r \in (\hat{L}_1)^m} (D_r(x)) = \bigcup_{\lambda \in [0,+\infty)} (B(\lambda x)) .$$

Definition (6.3.2-3):

The r-Expansion Path Set $D_r(x)$ has Linear Structure if and only if there exists a scalar $\theta\left(r,\lambda, \frac{x}{||x||}\right) \geqq 0$ such that

$$(6.3.2\text{-}5) \qquad B_r(\lambda x) = \theta(r,\lambda,x) \cdot B_r(x)$$

for $\lambda \in [0,+\infty)$.

Definition (6.3.2-4)

The Global Expansion Path Set $D(x)$ has linear structure if and only if there exists a scalar $\theta(\lambda,x)$ such that

$$(6.3.2\text{-}6) \qquad B(\lambda x) = \theta(\lambda,x) \cdot B(x)$$

for $\lambda \in [0,+\infty)$.

Similar to the common scaling law for global homothetic, semi-homogeneous and ray homothetic input structure, one has for an output structure $x \to \mathbb{P}(x)$ with these special structures, that

$$(6.3.2\text{-}7) \qquad \mathbb{P}(\lambda x) = \frac{F(\mathbb{H}(\lambda x))}{F(\mathbb{H}(x))} \cdot \mathbb{P}(x) , \quad \mathbb{P}(x) \neq \{0\} , \ \lambda \in [0,+\infty)$$

with

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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES. PART II.(U)

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$$F(\mathbb{H}(x)) : = (\mathbb{H}(x))^{\mathbb{D}\left(\frac{x}{||x||}\right)} \tag{6.3.2-8}$$

$$\mathbb{H}(x) : = ||x|| \tag{6.3.2-9}$$

in case $x \to \mathbb{P}(x)$ is semi-homogeneous.

Without proofs, since the parallel those given in Section (6.3-1), the following propositions apply.

<u>Proposition (6.3.2-1)</u>: (Färe and Shephard, 1977)

If the output structure $x \to \mathbb{P}(x)$ satisfies the scaling law (6.3.2-7) for $x \varepsilon (L_\infty)^n_+$, $\mathbb{P}(x) \neq \{0\}$, it has linear structure for the global expansion path sets $D(x)$.

The scalar $\theta(\lambda,x)$ for the global expansion path sets of $D(x)$ is

$$\theta(\lambda,x) = \frac{F(\mathbb{H}(\lambda x))}{F(\mathbb{H}(x))} \tag{6.3.2-10}$$

and in the case of semi-homogeneous output structure

$$\theta(\lambda,x) : = \frac{||\lambda x||^{\mathbb{D}\left(\frac{x}{||x||}\right)}}{||x||^{\mathbb{D}\left(\frac{x}{||x||}\right)}} = \lambda^{\mathbb{D}\left(\frac{x}{||x||}\right)} . \tag{6.3.2-11}$$

<u>Proposition (6.3.2-2)</u>: (Färe and Shephard, 1977)

If the output structure $x \to \mathbb{P}(x)$ has convex map sets $\mathbb{P}(x)$, $x \varepsilon (L_\infty)^n_+$, and axiom $\mathbb{P}$.6SS applies, linear structure of global expansion path sets $D(x)$ implies a scaling law of the form (6.3.2-7) in the weak* topology for $x \to \mathbb{P}(x)$.

In the case of ray homothetic output structures, the maximal revenue functional becomes

$$\mathbb{R}(x,r) = \frac{F(\mathbb{H}(x))}{F\left(\mathbb{H}\left(\frac{x}{||x||}\right)\right)} \cdot \mathbb{R}\left(\frac{x}{||x||}, r\right) , \quad \mathbb{P}(x) \neq \{0\} . \tag{6.3.2-12}$$

Hence, for any given input mix $\frac{x}{||x||}$, $\mathbb{R}\left(\frac{x}{||x||}, r\right)$ is a linear homogeneous price index functional for price histories r, when $\mathbb{R}(x,r) > 0$, $\mathbb{P}(x) \neq \{0\}$, and for scaling a vector x of input histories the scaling functional

$$\frac{F(\mathbb{H}(x))}{F\left(\mathbb{H}\left(\frac{x}{||x||}\right)\right)} \tag{6.3.2-13}$$

is obtainable as "price deflated" maximal revenue from x, where the price functional used as deflator depends upon the mix $\frac{x}{||x||}$ of input histories being scaled. As the vector x is scaled, the revenue maximizing vector $u^*\left(\frac{x}{||x||}\right)$ is scaled by the same functional (6.3.2-11). Thus the mix of output histories $u^*\left(\frac{x}{||x||}\right)$ is not altered.

REFERENCES

Al-Ayat, R. and R. Färe (1977), "On the Existence of Joint Production Functions," ORC 77-16, Operations Research Center, University of California, Berkeley.

Berge, C. (1963), TOPOLOGICAL SPACES, Macmillan Company, New York.

Bol, G. and O. Moeschlin (1975), "Isoquants of Continuous Production Correspondences," Naval Logistics Research Quarterly, Vol. 22, pp. 391-398.

Eichhorn, W. (1969), "Eine Verallgemeinerung des Begriffs der homogenen Produktionsfunktion," Unternehmensforschung, Vol. 13, pp. 99-109.

Eichhorn, W. (1978), FUNCTIONAL EQUATIONS IN ECONOMICS, Addison-Wesley, (forthcoming).

Färe, R. (1972), "Strong Limitationality of Essential Proper Subsets of Factors of Production," Zeitschrift für Nationalökonomie, Vol. 32, pp. 417-424.

Färe, R. and L. Jansson (1976), "Joint Inputs and the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 36, pp. 407-416.

Färe, R. and R. W. Shephard (1977), "Ray-Homothetic Production Functions," Econometrica, Vol. 45, pp. 133-146.

Shephard, R. W. (1953), COST AND PRODUCTION FUNCTIONS, Princeton University Press, Princeton.

Shephard, R. W. (1970:a), THEORY OF COST AND PRODUCTION FUNCTIONS, Princeton University Press, Princeton.

Shephard, R. W. (1970:b), "Proof of the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 30, pp. 7-34.

Shephard, R. W. (1974:b), "Semi-Homogeneous Production Functions," Lecture Notes in Economics and Mathematical Systems, Vol. 99, in PRODUCTION THEORY, Berlin.

Shephard, R. W. and R. Färe (1974), "The Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 34, pp. 69-90.